Tracking Relevance via Nested Sequents

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What makes a relevant logic relevant

- A logic L is said to have the *variable-sharing property* (VSP for short) if $A \rightarrow B$ is a theorem in L, $\mathrm{Var}(A) \cap \mathrm{Var}(B) \neq \varnothing$.
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- VSP is the intrinsic feature of relevance [Standefer, 2025]
- stronger forms of VSP: strong/signed VSP, DRP, strong/signed DRP

Proof-theory for relevant logic

- Existing modular calculi are display or labelled calculi provided in e.g.,
  display calculi: [Belnap, 1982, Goré, 1998]
  labelled calculi:
  [Viganò, 2000, Kurokawa and Negri, 2020, De Martin Polo, 2023]

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- Existing nested calculi excel at handling *binary* accessibility relations (e.g., modal logic, intuitionistic logic and their combinations). (e.g.,[Brünnler, 2009, Fitting, 2014, Balbiani et al., 2024])

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> Can we obtain nested calculi which are able to
> (1) **capture the ternary relation** of Routley-Meyer semantics;
> (2) **derive meta-logical properties** regarding variable-sharing?

In this work...

- we develop the first **nested sequent calculus** for B^+ which directly internalizes ternary relations
- with this calculus, we derive various **variable-sharing properties** syntactically: VSP, DRP (and their polarized variants)
- the proof-theoretic technique we use is inspired from **intuitionistic modal logic**, e.g., [Straßburger, 2013, Gao and Olivetti, 2026]

The relevant logic B^+

Routley-Meyer Relational Semantics

The **language** Frm is generated by:

$$\mathrm{Frm} \ni A := p \in \mathrm{At} \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A)$$

where At is a countable set of propositional atoms.

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A B^+**-frame** is a quadruple $\mathscr{F} = (\mathscr{O}, K, R, \leq)$ where

(i) K is a non-empty set of worlds

(ii) $\mathscr{O} \subseteq K$ is the set of *regular worlds*

(iii) $R \subseteq K \times K \times K$ is a ternary accessibility relation

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(iv) $\leq$ is a partial order on K, defined as $a \leq b \iff \exists u \in \mathscr{O}\ \&\ Ruab$

Moreover, for all $a, b, c, d \in K$:

- **(F1)** $a \leq a$
- **(F2)** If $a \leq b$ and $Rbcd$, then $Racd$.

Routley-Meyer Relational Semantics

A B^{+}**-model** $\mathscr{M} = (\mathscr{F}, V)$ pairs a frame with a valuation $V : \mathrm{At} \to \mathscr{P}(K)$ which is monotonic:

$$\text{if } x \in V(p) \text{ and } x \leq y, \text{ then } y \in V(p)$$

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Forcing Relation $\Vdash$

- $\mathscr{M}, x \Vdash p \iff x \in V(p)$
- $\mathscr{M}, x \Vdash B \wedge C \iff \mathscr{M}, x \Vdash B$ and $\mathscr{M}, x \Vdash C$
- $\mathscr{M}, x \Vdash B \vee C \iff \mathscr{M}, x \Vdash B$ or $\mathscr{M}, x \Vdash C$
- $\mathscr{M}, x \Vdash B \to C \iff \forall y, z \in K(Rxyz \;\&\; \mathscr{M}, y \Vdash B)$ then $\mathscr{M}, z \Vdash C$

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- $\mathscr{M}, x \Vdash B \vee C \iff \mathscr{M}, x \Vdash B$ or $\mathscr{M}, x \Vdash C$
- $\mathscr{M}, x \Vdash B \rightarrow C \iff \forall y, z \in K(Rxyz \;\&\; \mathscr{M}, y \Vdash B)$ then $\mathscr{M}, z \Vdash C$

Monotonicity property

For any formula $A \in \mathrm{Frm}$, if $\mathscr{M}, x \Vdash A$ and $x \leq y$, then $\mathscr{M}, y \Vdash A$.

The Hilbert system $\mathscr{H}_{B+}$ [Routley and Meyer, 1973]

ax1 $A \to A$

ax2 $(A \to B) \to ((B \to C) \to (A \to C))$

ax3 $(A \to (A \to B)) \to (A \to B)$

ax4 $(A_1 \wedge A_2) \to A_i$

ax5 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

ax6 $A_i \to (A_1 \vee A_2)$

ax7 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

ax8 $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

rules $\dfrac{A \quad A \to B}{B}$ $\qquad \dfrac{A \quad B}{A \wedge B}$ $\qquad \dfrac{A \to B \quad C \to D}{(B \to C) \to (A \to D)}$

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Theorem (Soundness and completeness)

A is derivable in $\mathscr{H}_{\mathrm{B}^+}$ if and only if A is B^+-valid.

A nested calculus for B^+

From sequent to nested sequent

- **simple sequent**:

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$S = A_1, \dots, A_k \Rightarrow B_1, \dots, B_j, [\,T_1\,], \dots, [\,T_m\,]$ [Brünnler, 2009]

$$tr(S): \quad \begin{tikzcd} tr(T_1) & \cdots & tr(T_m) \\ & A_1, \dots, A_k \Rightarrow B_1, \dots, B_k \arrow[ul, "{[\cdot]}"] \arrow[ur, "{[\cdot]}"'] & \end{tikzcd}$$

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- In ordinary nested sequents, a block usually refers to a single accessible world in Kripke semantics, e.g. R-successor in modal logic or $\leq$-upper world in intuitionistic logic.

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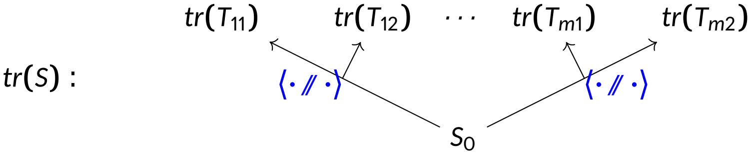
- However, for a ternary relation $Rxyz$, a world x sees a *pair* of worlds (y, z).
- **Our solution:** Using binary nested blocks $\langle \cdot /\!\!/ \cdot \rangle$



Syntax of nested sequents

Formulas are polarized by $\bullet$ (left/input) and $\circ$ (right/output)

- full sequent $\Gamma :: \Lambda, \Pi$
- left-sided $\Lambda :: \varnothing \mid A^\bullet \mid \langle \Lambda \sslash \Lambda \rangle \mid \Lambda, \Lambda$
- right-sided $\Pi :: A^\circ \mid \langle \Lambda \sslash \Gamma \rangle \mid \langle \Gamma \sslash \Lambda \rangle$

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Example

$\Sigma = p \vee q^\bullet, \langle p^\bullet \sslash p \vee q^\bullet \rangle, \langle q^\circ \sslash p \to q^\bullet \rangle$ is a full nested sequent,
$\Lambda = p \vee q^\bullet, \langle p^\bullet \sslash p \vee q^\bullet \rangle, \langle p^\bullet, q^\bullet \sslash p \to q^\bullet \rangle$ is a left-sided sequent.

Basic notions with nesting

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  $G\{\Sigma\}$ refers to a nested sequent which is fulfilled with Σ to $G\{\ \}$.

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  $G\{\Sigma\}$ refers to a nested sequent which is fulfilled with Σ to $G\{\ \}$.
- **pruning:** Σ^* means removing the unique output formula of Σ. By $G^*\{\ \}$ we mean removing the unique output formula in G.
- distinction between regular (non-nesting) and irregular (arbitrary nesting) worlds:

$$\overbrace{\Sigma}^{x\in\mathcal{O}}, \langle \overbrace{\Phi_1}^{a\in K} /\!\!/ \overbrace{\Phi_2}^{b\in K} \rangle \quad \text{and} \quad G\{ \overbrace{\Sigma}^{a}, \langle \overbrace{\Phi_1}^{b} /\!\!/ \overbrace{\Phi_2}^{c} \rangle \}$$

Example

$$\Phi = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} r \to s^\bullet \rangle, \langle q^\circ \mathbin{/\!\!/} p \to q^\bullet \rangle$$

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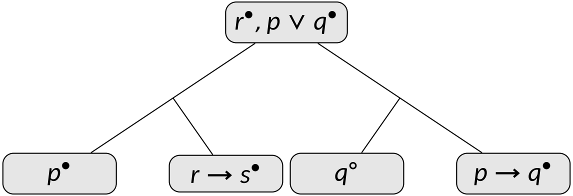

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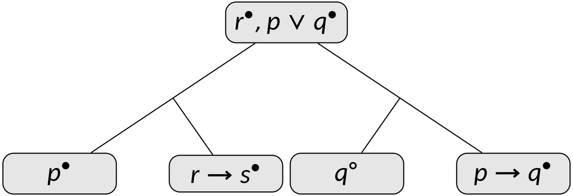

- $\Phi^* = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} r \rightarrow s^\bullet \rangle, \langle - \mathbin{/\!\!/} p \rightarrow q^\bullet \rangle$

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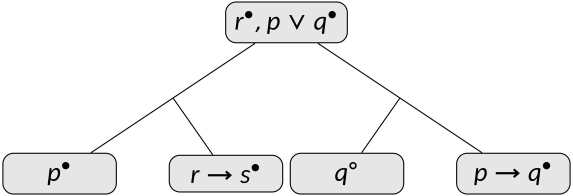

- $\Phi^* = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} r \to s^\bullet \rangle, \langle - \mathbin{/\!\!/} p \to q^\bullet \rangle$
- $\Phi = G\{r \to s^\bullet\}$ where $G\{\ \} = r^\bullet, p \vee q^\bullet, \langle p^\bullet \mathbin{/\!\!/} \{\ \} \rangle, \langle q^\circ \mathbin{/\!\!/} p \to q^\bullet \rangle$

The nested calculus $\mathbf{C}_{\mathrm{B}^+}$: basic rules

Initial sequents:

$$\frac{}{G\{\Sigma, p^{\bullet}, p^{\circ}\}}\ (\mathrm{id})$$

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Logical rules for $\wedge$, $\vee$:

$$\frac{G\{\Sigma, A^\bullet, B^\bullet\}}{G\{\Sigma, A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{G\{\Sigma, A^\circ\} \qquad G\{\Sigma, B^\circ\}}{G\{\Sigma, A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{G\{\Sigma, A^\bullet\} \qquad G\{\Sigma, B^\bullet\}}{G\{\Sigma, A \vee B^\bullet\}}\ \vee^\bullet \qquad \frac{G\{\Sigma, F^\circ\}}{G\{\Sigma, A \vee B^\circ\}}\ \vee^\circ, F \in \{A, B\}$$

The calculus $\mathbf{C}_{B^+}$: special rules

Logical rules for $\to$:

$$\frac{G\{\Sigma, \langle A^\bullet \mathbin{/\!\!/} B^\circ \rangle\}}{G\{\Sigma, A \to B^\circ\}} \to^\circ$$

The calculus $\mathbf{C}_{\mathrm{B}^+}$: special rules

Logical rules for $\rightarrow$:

$$\frac{G\{\Sigma, \langle A^\bullet /\!\!/ B^\circ\rangle\}}{G\{\Sigma, A \rightarrow B^\circ\}}\ \rightarrow^\circ$$

$$\frac{G^*\{\Sigma^*, A \rightarrow B^\bullet, \langle \Phi_1^*, A^\circ /\!\!/ \Phi_2^*\rangle\} \qquad G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 /\!\!/ \Phi_2, B^\bullet\rangle\}}{G\{\Sigma, A \rightarrow B^\bullet, \langle \Phi_1 /\!\!/ \Phi_2\rangle\}}\ \rightarrow^\bullet$$

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A structural rule (to mirror monotonicity of $\leq$):

$$\frac{\Sigma, \langle A^\bullet, \Phi_1 \mathbin{/\!\!/} A^\bullet, \Phi_2\rangle}{\Sigma, \langle A^\bullet, \Phi_1 \mathbin{/\!\!/} \Phi_2\rangle} \text{ lift}$$

Example

$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)^{\circ}$ is provable in $\mathbf{C}_{\mathrm{B}^+}$.

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Proof.

$$
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  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{}{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} A^\bullet, A^\circ\rangle}\,(\mathrm{id})}{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} A^\circ\rangle}\,\mathrm{lift}
        \quad
        \dfrac{\dfrac{}{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} B^\bullet, B^\circ\rangle}\,(\mathrm{id})}{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} B^\circ\rangle}\,\mathrm{lift}
      }{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} A \wedge B^\circ\rangle}\,\wedge^\circ
    }{\langle A^\bullet, B^\bullet \mathbin{/\!\!/} (A \wedge B) \vee (A \wedge C)^\circ\rangle}\,\vee^\circ
    \quad
    \dfrac{
      \dfrac{
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  }{
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$$

□

Two more structural rules (not in $\mathbf{C}_{B^+}$)

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$$\frac{\langle \Phi_1 \mathbin{/\!\!/} \Phi_2 \rangle}{G\{\Phi_1, \Phi_2\}}\ \mathrm{R1}$$

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- **Rule** R1 reflects (F1): $a \leq a$
- **Rule** R2 corresponds to (F2): If $a \leq b$ and $Rbcd$, then $Racd$

Compared with labelled calculi

Similar structural rules are necessarily included in a labelled calculus but do **NOT** need to be primitive here

Admissibility and invertibility

- A rule $\mathscr{R}$ is **admissible** if, when its premises are derivable at height n, the conclusion is derivable at height m. If $m \leq n$, the rule is **height-preserving admissible** (*hp-admissible*).

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Proposition

The rules R1 and R2 are hp-admissible in $\mathbf{C}_{B^+}$.

Semantic interpretation

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For a **left-sided sequent** $\Sigma = A_1^\bullet, \ldots, A_m^\bullet, \langle \Phi_{11} \mathbin{/\!\!/} \Phi_{12} \rangle, \ldots, \langle \Phi_{j1} \mathbin{/\!\!/} \Phi_{j2} \rangle$,
$x \Vdash \Sigma$ iff:

- for any $i \in \{1, \ldots, m\}$, $x \Vdash A_i^\bullet$; and
- for any $k \in \{1, \ldots, j\}$, there exist y, z with $Rxyz$, $y \Vdash \Phi_{k1}$ and $z \Vdash \Phi_{k2}$

Semantic interpretation

Let $\mathscr{M} = (\mathscr{O}, K, R, V)$ be a B^+-model and $x, y, z \in K$.

For a **left-sided sequent** $\Sigma = A_1^\bullet, \ldots, A_m^\bullet, \langle \Phi_{11} /\!\!/ \Phi_{12} \rangle, \ldots, \langle \Phi_{j1} /\!\!/ \Phi_{j2} \rangle$, $x \Vdash \Sigma$ iff:

- for any $i \in \{1, \ldots, m\}$, $x \Vdash A_i^\bullet$; and
- for any $k \in \{1, \ldots, j\}$, there exist y, z with $Rxyz$, $y \Vdash \Phi_{k1}$ and $z \Vdash \Phi_{k2}$

For a **full sequent**,

- $x \Vdash \Sigma, A^\circ$ iff $x \Vdash \Sigma$ implies $x \Vdash A$
- $x \Vdash \Sigma, \langle \Phi_1 /\!\!/ \Phi_2 \rangle$ (where Φ_1 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $z \Vdash \Phi_2$ implies $y \Vdash \Phi_1$
- $x \Vdash \Sigma, \langle \Phi_1 /\!\!/ \Phi_2 \rangle$ (where Φ_2 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $y \Vdash \Phi_1$ implies $z \Vdash \Phi_2$

Remark on semantic interpretation

The satisfaction of a full sequent says if the left-sided part is satisfied, so is the unique output.

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According to the definition, for a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$, if $\mathscr{M}, w \nVdash G\{\Sigma, A^\circ\}$, then there exists some $x \in K$ such that $x \nVdash \Sigma, A^\circ$.

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The satisfaction of a full sequent says if the left-sided part is satisfied, so is the unique output.

According to the definition, for a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$, if $\mathscr{M}, w \not\Vdash G\{\Sigma, A^\circ\}$, then there exists some $x \in K$ such that $x \not\Vdash \Sigma, A^\circ$.

Notice that a block is interpreted differently depending on whether it contains the output formula and where that formula is precisely located:

- if a block **does not** contain the unique output formula, it is interpreted **existentially**
- if a block **does** contain the unique output formula, it is interpreted **universally**

Similar practice for interpreting nested sequents is also used in intuitionistic modal logic, e.g. [Straßburger, 2013, Gao and Olivetti, 2026]

Soundness of $\mathbf{C}_{\mathrm{B}^+}$

For a rule (r) in $\mathbf{C}_{\mathrm{B}^+}$, we say (r) is *valid* if whenever all of its premises are valid, the conclusion is also valid.

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All the rules in $\mathbf{C}_{B^+}$ are B^+-valid.

Theorem (Soundness of $\mathbf{C}_{B^+}$)

If a nested sequent Σ is provable in $\mathbf{C}_{B^+}$, then Σ is B^+-valid.

A syntactic proof of completeness of $\mathbf{C}_{B^+}$

The cut rule can be applied to any component of the sequent:

$$\frac{G^*\{\Sigma^*, A^\circ\} \qquad G\{\Sigma, A^\bullet\}}{G\{\Sigma\}}\ \text{cut}$$

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Theorem (Completeness of $\mathbf{C}_{B^+}$ + cut)

If a formula $A \in \mathrm{Frm}$ is valid, then it is provable in $\mathbf{C}_{B^+}$ + cut.

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If a formula $A \in \mathrm{Frm}$ is valid, then it is provable in $\mathbf{C}_{\mathrm{B}^+} + \mathrm{cut}$.

Theorem (Cut-admissibility)

The cut rule is admissible in $\mathbf{C}_{\mathrm{B}^+}$.

How C_{B^+} captures relevance flow internally

Setup: some provable equivalences

Let Φ be a multi-set of $\bullet$-formulas and Ψ a nested sequent. Then

$$\langle \Phi \mathbin{/\!\!/} \Psi \rangle \quad \Longleftrightarrow \quad \langle - \mathbin{/\!\!/} \Phi, \Psi \rangle$$

We call a nested sequent in the form $\langle - \mathbin{/\!\!/} \Phi \rangle$ **0-free** sequent.

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Consequently, for a formula $A \to B$:

$$A \to B^{\circ} \quad \Leftrightarrow \quad \langle A^{\bullet} \mathbin{/\!\!/} B^{\circ} \rangle \quad \Leftrightarrow \quad \langle - \mathbin{/\!\!/} A^{\bullet}, B^{\circ} \rangle$$

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$$A \rightarrow B^{\circ} \quad \Leftrightarrow \quad \langle A^{\bullet} \mathbin{/\!\!/} B^{\circ} \rangle \quad \Leftrightarrow \quad \langle - \mathbin{/\!\!/} A^{\bullet}, B^{\circ} \rangle$$

We only need to focus on 0-free sequents for studying meta-logical properties.

Reachability

Let Σ be a nested sequent. If

$$\Sigma = G\{\Phi\} \quad \text{and} \quad \Phi = \Phi', \langle \Psi_1 /\!\!/ \Psi_2 \rangle \quad \text{for some } G\{\ \}$$

then we say Ψ_1 and Ψ_2 directly belong to Φ (in Σ), written as $\Psi_i \in_0^+ \Psi$.

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For example, let $\Sigma = p \vee q^\bullet, \langle p^\bullet, \langle q^\bullet \mathbin{/\!\!/} p^\circ \rangle \mathbin{/\!\!/} t^\bullet \rangle$ and denote

$$\Phi_1 = p^\bullet, \langle q^\bullet \mathbin{/\!\!/} p^\circ \rangle \qquad \Phi_2 = t^\bullet \qquad \Phi_3 = p^\bullet \qquad \Phi_4 = p^\circ$$

we have $\Phi_i \in_0^+ \Phi_1 \in_0^+ \Sigma$ for $i \in \{3, 4\}$ and $\Phi_2 \in^+ \Sigma$ but not $\Phi_1 \in_0^+ \Phi_2$.

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Definition (Reachable nodes)

Let Σ be a nested sequent and $\Phi, \Psi \in^+ \Sigma$. We say $\widetilde{\Phi}$ is *reachable* to $\widetilde{\Psi}$ if $\Phi \in^+ \Psi$ or $\Psi \in^+ \Phi$.

Variable-sharing property

Lemma

Let $\langle - \mathbin{/\!\!/} \Phi \rangle$ be a nested sequent with the output formula B. If it is provable, then there exist $\Sigma, \Psi \in^{+} \Phi$ such that

- $B^{\circ} \in \widetilde{\Psi}$;
- $\widetilde{\Sigma}$ *is reachable to* $\widetilde{\Psi}$;
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Theorem (VSP for $\mathbf{C}_{\mathrm{B+}}$)

If $\langle - /\!\!/ A^{\bullet}, B^{\circ} \rangle$ is provable in $\mathbf{C}_{\mathrm{B+}}$, then $\mathrm{Var}(A) \cap \mathrm{Var}(B) \neq \varnothing$.

The case of $\to^\bullet$

Consider an application of $\to^\bullet$ in the following form:

$$\frac{\Gamma_1 = G^*\{\Sigma^*, C \to D^\bullet, \langle \Phi_1^*, C^\circ \mathbin{/\!\!/} \Phi_2^* \rangle\} \qquad \Gamma_2 = G\{\Sigma, C \to D^\bullet, \langle \Phi_1 \mathbin{/\!\!/} \Phi_2, D^\bullet \rangle\}}{\Gamma = G\{\Sigma, C \to D^\bullet, \langle \Phi_1 \mathbin{/\!\!/} \Phi_2 \rangle\}}$$

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Assume for Γ_2, the output node is Θ and the output formula is θ.

- if the required $\rho^\bullet$ is not the displayed $D^\bullet$, apply IH and take the same ρ for Γ.
- if the required $\rho^\bullet$ for Γ_2 is $D^\bullet$. This means Θ is reachable to $\widetilde{\Phi_2}, D^\bullet$. By the tree structure, Θ is also reachable to $\widetilde{\Sigma}, C \rightarrow D^\bullet$. Take $C \rightarrow D^\bullet$ as ρ for Γ, by IH, $\mathrm{Var}(\theta) \cap \mathrm{Var}(D) \neq \varnothing$, so $\mathrm{Var}(\theta) \cap \mathrm{Var}(C \rightarrow D) \neq \varnothing$.

The case of $\rightarrow^\circ$

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$$\frac{\Gamma' = G\{\Sigma, \langle C^\bullet \mathbin{/\!\!/} D^\circ \rangle\}}{\Gamma = G\{\Sigma, C \rightarrow D^\circ\}}$$

D° is θ° for Γ' in the statement of the lemma; by IH, $\mathrm{Var}(\rho) \cap \mathrm{Var}(D) \neq \varnothing$.
Take the same ρ for Γ, it follows directly $\mathrm{Var}(\rho) \cap \mathrm{Var}(C \rightarrow D) \neq \varnothing$.

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By reachability, $\rho^{\bullet}$ occurs either in $\widetilde{\Sigma}$ or somewhere deep in Σ or $G\{\ \}$:

- $\widetilde{\Sigma}, C \rightarrow D^{\circ}$ is reachable to itself;
- any other node in Σ or $G\{\ \}$ that is reachable to the output node D° in the premise is also reachable to $\widetilde{\Sigma}, C \rightarrow D^{\circ}$ in the conclusion.

Depth of formula occurrences

Let F be a formula and $\hat{A}$ occurs in F, we define the depth of $\hat{A}$ in F, denoted $\mathrm{dp}_F(\hat{A})$, as follows:

- $\mathrm{dp}_F(\hat{F}) := 0$
- $\mathrm{dp}_F(\hat{B}) := \mathrm{dp}_F(\hat{C}) = n$, if they are direct sub-occurrences of $\widehat{B@C}$ and $\mathrm{dp}_F(\widehat{B@C}) = n$ where $@ \in \{\vee, \wedge\}$
- $\mathrm{dp}_F(\hat{B}) = \mathrm{dp}_F(\hat{C}) = n + 1$ if they are direct sub-occurrences of $\widehat{B \to C}$ and $\mathrm{dp}_F(\widehat{B \to C}) = n$

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For two formula occurrences $\hat{A}, \hat{G}$ in F such that $\hat{A}$ occurs in $\hat{G}$, we write $\mathrm{dp}_G(\hat{A})$ for the depth of $\hat{A}$ in $\hat{G}$ regarding G as an independent formula.

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Example

Let $F = p \to (p \to ((s \land t) \lor q))$ and $G = p \to ((s \land t) \lor q)$. By definition $\mathrm{dp}_F(\hat{t}) = 2$, $\mathrm{dp}_G(\hat{t}) = 1$ and $\mathrm{dp}_F(\hat{G}) = 1$.

Depth of formula occurrences in a sequent

Let Σ be a sequent. For any $\Phi \in^+ \Sigma$, we define $\mathrm{dp}_\Sigma(\Phi)$ as follows:

- $\mathrm{dp}_\Sigma(\Sigma) = 0$
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For any $A \in \widetilde{\Phi}$ for some $\Phi \in^+ \Sigma$, we define $\mathrm{dp}_\Sigma(A) := \mathrm{dp}_\Sigma(\Phi)$.

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For any $A \in \widetilde{\Phi}$ for some $\Phi \in^+ \Sigma$, we define $\mathrm{dp}_\Sigma(A) := \mathrm{dp}_\Sigma(\Phi)$. Furthermore, for any formula occurrences $\hat{B}$ in A, we define its depth in Σ with respect to A as $\mathrm{dp}^A_\Sigma(\hat{B}) := \mathrm{dp}_\Sigma(A) + \mathrm{dp}_A(\hat{B})$.

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Intuition about the rules

In the case of $\rightarrow^\bullet$ and $\rightarrow^\circ$, the change in depth at a given node is exactly the opposite of the change in depth in the formula; therefore, the depth of the shared variable as the witness remains unchanged.

Depth relevance property

Lemma

Let $\langle - \mathbin{/\!\!/} \Sigma \rangle$ be a 0-free sequent with the output formula θ. If it is provable, then there exist $\Phi, \Psi \in^{+} \Sigma$ such that

(a). $\theta^{\circ} \in \widetilde{\Phi}$;

(b). $\widetilde{\Psi}$ *is reachable to* $\widetilde{\Phi}$;

(c). *there exists* $\chi^{\bullet} \in \widetilde{\Psi}$ *and a variable* p *such that* $p \in \mathrm{Var}(\theta) \cap \mathrm{Var}(\chi)$ *and there are two occurrences* $\hat{p}_1, \hat{p}_2$ *of* p *such that* $\mathrm{dp}^{\theta}_{\Sigma}(\hat{p}_1) = \mathrm{dp}^{\chi}_{\Sigma}(\hat{p}_2)$.

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Theorem (DRP for B^{+})

If $A \to B$ is a theorem in B^{+}, then there exists a variable p such that $p \in \mathrm{Var}(A) \cap \mathrm{Var}(B)$ and there are two occurrences $\hat{p}_1, \hat{p}_2$ of p in A and B respectively with $\mathrm{dp}_A(\hat{p}_1) = \mathrm{dp}_B(\hat{p}_2)$.

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Consider an application of $\to^\bullet$ in the following form:

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By IH, there is $\hat{p}_2$ in D° in Γ' satisfying the condition.

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Consider an application of $\longrightarrow^{\bullet}$ in the following form:

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By IH, there is $\hat{p}_2$ in D° in Γ' satisfying the condition.
We take the corresponding occurrence $\hat{p}_2'$ of $\hat{p}_2$ in $\hat{D}$ in $C \longrightarrow D^{\circ}$.
It suffices to show $\mathrm{dp}_{\Gamma}^{C \rightarrow D}(\hat{p}_2') = \mathrm{dp}_{\Gamma'}^{D}(\hat{p}_2)$.

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By definition,
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We take the corresponding occurrence $\hat{p}'_2$ of $\hat{p}_2$ in $\hat{D}$ in $C \longrightarrow D^\circ$.
It suffices to show $\mathrm{dp}^{C \to D}_{\Gamma}(\hat{p}'_2) = \mathrm{dp}^{D}_{\Gamma'}(\hat{p}_2)$.

By definition,
$\mathrm{dp}^{C \to D}_{\Gamma}(\hat{p}'_2) = \mathrm{dp}_{\Gamma}(\Sigma, C \longrightarrow D^\circ) + \mathrm{dp}_{C \to D}(\hat{p}'_2)$ and
$\mathrm{dp}^{D}_{\Gamma'}(\hat{p}_2) = \mathrm{dp}_{\Gamma'}(\Sigma, \langle C^\bullet \mathbin{/\!\!/} D^\circ \rangle) + \mathrm{dp}_{D}(\hat{p}_2)$.

$\mathrm{dp}_{\Gamma}(\Sigma, C \longrightarrow D^\circ) = \mathrm{dp}_{\Gamma'}(\Sigma, D^\circ) - 1$ and $\mathrm{dp}_{C \to D}(\hat{p}'_2) = \mathrm{dp}_{D}(\hat{p}_2) + 1$, so the equality holds as required.

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Second, the proofs are **modular**, in the sense:

Reachability+	polarity+	depth	$\Rightarrow$	strong DRP
Reachability+		depth	$\Rightarrow$	DRP
Reachability+	polarity		$\Rightarrow$	strong VSP
Reachability			$\Rightarrow$	VSP

Conclusion

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Nested calculus provides a syntactic account of relevance

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Example

Let $\Sigma_1 = p^\bullet, \langle q^\circ \sslash p^\bullet, \langle q^\bullet \sslash q^\bullet \rangle\rangle$ and $\Sigma_2 = p^\bullet, \langle q^\bullet \sslash p^\bullet, \langle q^\bullet \sslash q^\circ \rangle\rangle$. By definition, Σ_1 is not a focused sequent but Σ_2 is.

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General proof idea towards ... CIP?

- to show the 0-free focused fragment of $\mathbf{C}_{B^+}$ is adequate
- to show the 'active' part of a focused sequent gives a unique split which is adequate for Meahara's induction of interpolation for sequents

Further topics

Other obtained results (not included in the slides):

- extended the proof-theoretic framework to a modal logic $B^+K^\Box$
- simulated $\mathbf{C}_{B^+}$ by a labelled calculus

Further topics

Other obtained results (not included in the slides):

- extended the proof-theoretic framework to a modal logic $B^+K^\square$
- simulated $\mathbf{C}_{B^+}$ by a labelled calculus

Ongoing work:

- treat **B** with De Morgan negation: one more structural operator for $\sim$, in a bi-nested flavor
- treat stronger logics such as $\mathbf{TW}^+$, $\mathbf{RW}^+$: consider other structural rules

Further topics

Other obtained results (not included in the slides):

- extended the proof-theoretic framework to a modal logic $\mathsf{B}^{+}\mathsf{K}^{\Box}$
- simulated $\mathbf{C}_{\mathsf{B}^{+}}$ by a labelled calculus

Ongoing work:

- treat $\mathbf{B}$ with De Morgan negation: one more structural operator for $\sim$, in a bi-nested flavor
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Thank you for your attention!

Details are found at: `https://ghlogic123.github.io/nested_RL.pdf`

References I

[Balbiani et al., 2024] Balbiani, P., Gao, H., Gencer, Ç., and Olivetti, N. (2024).
A Natural Intuitionistic Modal Logic: Axiomatization and Bi-Nested Calculus.
In *32nd EACSL Annual Conference on Computer Science Logic (CSL 2024)*, volume 288, pages 13:1–13:21.

[Belnap, 1982] Belnap, N. D. (1982).
Display logic.
Journal of philosophical logic, pages 375–417.

[Brünnler, 2009] Brünnler, K. (2009).
Deep sequent systems for modal logic.
Arch. Math. Log., 48(6):551–577.

[De Martin Polo, 2023] De Martin Polo, F. (2023).
Modular labelled calculi for relevant logics.
The Australasian Journal of Logic, 20(1):47–87.

References II

[Fitting, 2014] Fitting, M. (2014).
Nested sequents for intuitionistic logics.
Notre Dame J. Formal Log., 55(1):41–61.

[Gao and Olivetti, 2026] Gao, H. and Olivetti, N. (2026).
Taming complexity in intuitionistic modal logic: The case of FIK and its shallow calculus.
In *Proceedings of the Sixteenth International Conference on Advances in Modal Logic, Amsterdam, The Netherlands, 29-06-2026*, volume 447 of *Electronic Proceedings in Theoretical Computer Science*, pages 407–426. Open Publishing Association.

[Goré, 1998] Goré, R. (1998).
Substructural logics on display.
Logic Journal of IGPL, 6(3):451–504.

References III

[Kurokawa and Negri, 2020] Kurokawa, H. and Negri, S. (2020).
Relevant logics: from semantics to proof systems.
In *Contemporary Logic and Computing*, pages 259–306. College publications.

[Routley and Meyer, 1973] Routley, R. and Meyer, R. K. (1973).
The semantics of entailment i.
In Leblanc, H., editor, *Truth, Syntax, and Modality (Proceedings Of The Temple University Conference on Alternative Semantics)*, pages 199–243. North-Holland Publishing Company.

[Standefer, 2025] Standefer, S. (2025).
Variable-sharing as relevance.
In *New directions in relevant logic*, pages 97–117. Springer.

References IV

[Straßburger, 2013] Straßburger, L. (2013).
Cut elimination in nested sequents for intuitionistic modal logics.
In *Foundations of Software Science and Computation Structures: 16th International Conference, FOSSACS 2013*, pages 209–224. Springer.

[Viganò, 2000] Viganò, L. (2000).
Labelled Non-classical Logics.
Springer.