A Nested Approach to Relevant Logic (Technical Report)

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1 The negation-free relevant logic B^+

In this section, we introduce a Hilbert-style proof system together with a relational semantics for the negation-free fragment of the relevant logic B (cf. [28]), denoted B^+.

We begin by defining the propositional language considered throughout this paper.

Definition 1.1. Let At be a countable set of propositional atoms, ranged over by $p, q, \ldots$. The propositional language is generated by the set of connectives $\mathsf{Con} = \{\wedge, \vee, \rightarrow\}$. We denote by Frm the set of propositional formulas, which are inductively generated by the following grammar:

$$A ::= p \in \mathsf{At} \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A)$$

The *complexity* of a formula A, denoted by $|A|$, is defined as follows: $|p| := 0$ (for every $p \in \mathsf{At}$) and $|A \heartsuit B| := \max(|A|, |B|) + 1$ (for $\heartsuit \in \{\wedge, \vee, \rightarrow\}$). Moreover, we denote the set of propositional atoms occurring in A by $\mathsf{Var}(A)$. For a set Γ of formulas, let $\mathsf{Var}(\Gamma) := \bigcup_{A \in \Gamma} \mathsf{Var}(A)$.

Next, we examine B^+, presenting its Hilbert system, denoted by $\mathscr{H}_{\mathsf{B}^+}$, together with the associated relational semantics. The former is displayed in Figure 1. As for the latter, we present the Routley–Meyer semantics for B^+ which features a ternary relation among states [27, 28, 23].

ax1	$A \rightarrow A$
ax2	$A_1 \wedge A_2 \rightarrow A_i$
ax3	$(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))$
ax4	$A_i \rightarrow (A_1 \vee A_2)$ where $i \in \{1,2\}$
ax5	$(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$
ax6	$A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$

$$\frac{A \quad A \rightarrow B}{B}\ \mathsf{mp} \qquad \frac{A \quad B}{A \wedge B}\ \mathsf{adj} \qquad \frac{A \rightarrow B \quad C \rightarrow D}{(B \rightarrow C) \rightarrow (A \rightarrow D)}\ \mathsf{aff}$$

Figure 1: The Hilbert system $\mathscr{H}_{\mathsf{B}^+}$ for B^+.

Remark 1.2. Before turning to the formal details, it is worth noting that relational semantics for propositional relevant logics admit several variants. One may, for instance, distinguish a subset of *regular states* within the set of all states, or alternatively collapse this structure to a single distinguished regular state, typically denoted by 0 (cf. e.g., [26, 15, 32]).

Definition 1.3 (B^+-frames and B^+-models)**.** A B^+*-frame* is a triple $\mathscr{F} = \langle K, \mathscr{O}, R \rangle$, where K is a non-empty set of worlds, $\mathscr{O} \subseteq K$ is a non-empty set of *regular worlds*, and R is a ternary relation on K. For any $a, b \in K$, we define a binary relation $\leq$ as follows: $a \leq b$ iff $\exists x (x \in \mathscr{O}$ & $Rxab)$. The following conditions are imposed on B^+-frames. For all $a, b, c, d \in K$:

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(F1) $a \leq a$
(F2) if $a \leq b$ and $Rbcd$, then $Racd$.

A B^+-*model* is a pair $\mathscr{M} = \langle \mathscr{F}, V \rangle$, where $\mathscr{F}$ is a B^+-frame and $V : \mathsf{At} \longrightarrow \wp(K)$ is a valuation function satisfying the following condition: for all $p \in \mathsf{At}$ and all $a, b \in K$, if $a \leq b$ and $a \in V(p)$, then $b \in V(p)$.

Definition 1.4. The satisfaction relation in B^+ for propositional formulas is defined inductively in the following way. For all $a \in K$ and $A, B \in \mathsf{Frm}$:

$$\mathscr{M}, a \Vdash p \quad \text{iff} \quad a \in V(p), \text{ for } p \in \mathsf{At} \tag{i}$$

$$\mathscr{M}, a \Vdash A \wedge B \quad \text{iff} \quad \mathscr{M}, a \Vdash A \text{ and } \mathscr{M}, a \Vdash B \tag{ii}$$

$$\mathscr{M}, a \Vdash A \vee B \quad \text{iff} \quad \mathscr{M}, a \Vdash A \text{ or } \mathscr{M}, a \Vdash B \tag{iii}$$

$$\mathscr{M}, a \Vdash A \to B \quad \text{iff} \quad \forall b, c \in K, \text{ if } Rabc \text{ and } \mathscr{M}, b \Vdash A, \text{ then } \mathscr{M}, c \Vdash B \tag{iv}$$

A formula A is *satisfied* in $\mathscr{M}$ iff $\mathscr{M}, x \Vdash A$ for all $x \in \mathscr{O}$. A formula A is *valid* in a frame $\mathscr{F}$ iff it is satisfied in every B^+-model based on $\mathscr{F}$. A formula A is B^+-*valid* iff it is valid in every B^+-frame $\mathscr{F}$.

Lemma 1.5 ([27, 28])**.** *For all $a, b \in K$ and $A \in \mathsf{Frm}$, if $a \leq b$ and $a \Vdash A$, then $b \Vdash A$.*

Theorem 1.6 ([27, 28])**.** *A is a derivable in $\mathscr{H}_{\mathsf{B}^+}$ if and only if A is B^+-valid.*

Lemma 1.7 (Verification Lemma)**.** *Let $\mathscr{M}$ be a B^+-model. For all $w \in K$, $w \Vdash A$ implies $w \Vdash B$ iff for all $x \in \mathscr{O}$, $x \Vdash A \to B$.*

2 Nested sequents for relevant implication

In this section, we develop the proof theory of B^+ by extending the standard nested framework so that it can capture the ternary relation of Routley-Meyer frames. Based on this novel notion of nested sequents, we introduce a nested calculus for B^+ built on *binary nested blocks*, i.e., blocks containing two arguments. We then present the semantic interpretation of the proposed framework and prove soundness of the calculus. Finally, we establish completeness by means of a syntactic proof of (cut)-admissibility.

First, we define the syntax of nested sequents based on Frm where formulas are decorated with polarities $\bullet$ and $\circ$. We make use of a structural operator $\langle \cdot \mathbin{/\!/} \cdot \rangle$ with two arguments (also called component) to generate the nesting structure and this binary operator is called a *relevant block*.

Definition 2.1. Nested sequents are generated by the following grammar:

$$\Gamma :: \Lambda, \Pi \qquad \Lambda :: A_1^\bullet, \ldots, A_n^\bullet, \langle \Lambda_{11} \mathbin{/\!/} \Lambda_{12} \rangle, \ldots, \langle \Lambda_{k1} \mathbin{/\!/} \Lambda_{k2} \rangle \qquad \Pi :: A^\circ \mid \langle \Lambda \mathbin{/\!/} \Gamma \rangle \mid \langle \Gamma \mathbin{/\!/} \Lambda \rangle$$

Expressions in the above definition containing an output formula A° are called *full* nested sequent (sequent for short) while those with only $\bullet$-formulas occurring nestedly are called *left-sided* sequents. According to the definition, a sequent consists of two parts, the left part which only contains $\bullet$-decorated formulas, and the right part which is a single formula A° or a relevant block where exactly one of the two arguments is a full sequent. We also need the notion of *generic sequents* when it is not necessary to specify whether it is full or left-sided. To make it clear, we shall use the following naming for different types of sequents in this paper:

$$\text{full sequents: } \Gamma \qquad \text{left-sided sequents: } \Lambda \qquad \text{generic sequents: } \Sigma, \Phi, \Psi$$

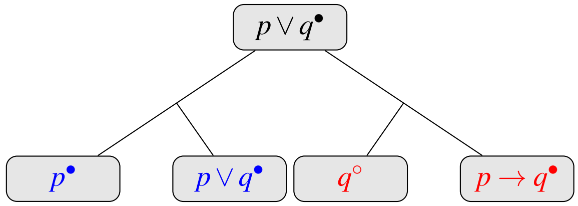

Figure 2: Graphic representation of $p \vee q^\bullet, \langle p^\bullet /\!\!/ p \vee q^\bullet \rangle, \langle q^\circ /\!\!/ p \to q^\bullet \rangle$

Remark 2.2. To the best of our knowledge, binary nested sequents representing trees of pairs of sequents have not been considered in the literature. In [1], a binary structure $[A : S]$ is considered, where A and S go in pairs and it looks like our binary nested block. However, in their definition, A is a decorated formula which is neither involved in the nested structure, nor decomposed by rule applications, whereas S is a nested sequent. Therefore, this kind of modified nested sequents still lies in the category of so-called nested sequents with unary blocks.

As usual in nested calculi, we have the notion of context in order to specify where the rule is applied within the nested structure. We follow the terminology and notation used in [4], that is, by $G\{\ \}$, we mean a nested sequent holding an empty component with the unique palceholder $\{\ \}$ and $G\{\Sigma\}$ refers to a nested sequent which is fulfilled with Σ to $G\{\ \}$. [1] This is slightly different from the initial use of context in e.g. [34, 7].

We write $\langle \Phi /\!\!/ - \rangle$ and $\langle - /\!\!/ \Phi \rangle$ if either component in the relevant pair is empty. Since a full nested sequent only contains one output formula and our calculus will only operate on full sequents, to simplify notations, we will generically use $\Sigma, \Phi, \Psi, \ldots$ to denote both sequents and left-sided sequents. We define two operators for a (left-sided) sequent Σ: (1) $\widetilde{\Sigma}$ to remove all the possible blocks in Σ; (2) Σ^* to remove the unique output formula. Notice that if Σ is a left-sided sequent, then $\Sigma^* = \Sigma$. The $*$-operator is adopted to context as well, by $G^*\{\ \}$ we mean removing the unique output formula in G.

Example 2.3. By definition, $\Sigma = p \vee q^\bullet, \langle p^\bullet /\!\!/ p \vee q^\bullet \rangle, \langle q^\circ /\!\!/ p \to q^\bullet \rangle$ is a full nested sequent, $\Sigma^* = p \vee q^\bullet, \langle p^\bullet /\!\!/ p \vee q^\bullet \rangle, \langle - /\!\!/ p \to q^\bullet \rangle$ and $\widetilde{\Sigma} = p \vee q^\bullet$. A graphic representation of Σ is found in Figure 2.

A nested sequent calculus for the logic B^+ is given in Figure 3. We provide some explanations for these rules as below.

Remark 2.4. In $\mathbf{C}_{\mathsf{B}^+}$ all logical rules can be applied at any component within any nested block while (lift) is only applicable to blocks belonging to the top-level component. This distinction is useful for separating the semantic roles of the relations $a \le b$ and $Rabc$ within the calculus. As will be shown later in the semantic interpretation, the top-level component of a nested sequent is associated with the regular worlds of the semantics, while the relevant block immediately succeeding it corresponds to two worlds, say $a, b \in K$, to be read as $a \le b$. By contrast, the components occurring inside nested sequents are associated with *irregular* worlds in K. Thus $Rxab$ (where $x \in \mathcal{O}$) and $Rabc$ are represented graphically in a nested sequent as

$$\overbrace{\Sigma}^{x\in\mathcal{O}}, \langle \overbrace{\Phi_1}^{a\in K} /\!\!/ \overbrace{\Phi_2}^{b\in K} \rangle \quad \text{and} \quad G\{\overbrace{\Sigma}^{a}, \langle \overbrace{\Phi_1}^{b} /\!\!/ \overbrace{\Phi_2}^{c} \rangle\}$$

The rules $(\mathsf{lift}), \to^\bullet, \to^\circ$ involving propagation or production of relevant blocks make the two arguments contained in the same block not interchangeable. The rule (lift) – some variants thereof being familiar from intuitionistic logic (c.f. [12]) – expresses the monotonicity property semantically captured by

[1] Note that Σ can be either full or left-sided according to whether $G\{\ \}$ contains an output formula.

Initial sequents:

$$\frac{}{G\{\Sigma, p^\bullet, p^\circ\}}\ (\mathsf{id})$$

Logical rules:

$$\frac{G\{\Sigma, A^\bullet, B^\bullet\}}{G\{\Sigma, A\wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{G\{\Sigma, A^\circ\} \quad G\{\Sigma, B^\circ\}}{G\{\Sigma, A\wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{G\{\Sigma, A^\bullet\} \quad G\{\Sigma, B^\bullet\}}{G\{\Sigma, A\vee B^\bullet\}}\ \vee^\bullet \qquad \frac{G\{\Sigma, A^\circ\}}{G\{\Sigma, A\vee B^\circ\}}\ \vee^\circ \qquad \frac{G\{\Sigma, B^\circ\}}{G\{\Sigma, A\vee B^\circ\}}\ \vee^\circ$$

$$\frac{G^*\{\Sigma^*, A\to B^\bullet, \langle \Phi_1^*, A^\circ \mathbin{/\!/} \Phi_2^*\rangle\} \quad G\{\Sigma, A\to B^\bullet, \langle \Phi_1 \mathbin{/\!/} \Phi_2, B^\bullet\rangle\}}{G\{\Sigma, A\to B^\bullet, \langle \Phi_1 \mathbin{/\!/} \Phi_2\rangle\}}\ \to^\bullet \qquad \frac{G\{\Sigma, \langle A^\bullet \mathbin{/\!/} B^\circ\rangle\}}{G\{\Sigma, A\to B^\circ\}}\ \to^\circ$$

Structural rules:

$$\frac{\Sigma, \langle A^\bullet, \Phi_1 \mathbin{/\!/} A^\bullet, \Phi_2\rangle}{\Sigma, \langle A^\bullet, \Phi_1 \mathbin{/\!/} \Phi_2\rangle}\ (\mathsf{lift})$$

Figure 3: $\mathbf{C}_{\mathsf{B}^+}$, a nested sequent calculus for B^+

Lemma 1.5. The rules for $\to$ are inspired by the semantic clause for implication. From a backward view, $\to^\circ$ states that an implication is false at a world a when there exist worlds b, c with $Rabc$, where b satisfies the antecedent and c falsifies the consequent. Syntactically, this is captured by the relevant block $G\{\Sigma, \langle A^\bullet \mathbin{/\!/} B^\circ\rangle\}$ in the premise, whose two components represent b and c. Dually, $\to^\bullet$ reflects the semantic use of already opened worlds: any relevant block associated with a sequent containing $A \to B^\bullet$ may be used. The two premises then represent the worlds b and c, with A° in the first component and $B^\bullet$ in the second. To keep the single output structure, we have the similar 'pruning' for the left premise as [34] to remove the existing output formula and put A° as the unique output formula.

As usual, a *derivation* is a rooted tree defined in the standard way, and its *height* is the length of its longest branch. We say that a formula A is $\mathbf{C}_{\mathsf{B}^+}$*-derivable* whenever the sequent A° is $\mathbf{C}_{\mathsf{B}^+}$-derivable.

Definition 2.5 (Formula occurrences)**.** We say an occurrence of a formula occurring in the conclusion of an application of a logical rule (r) is *side* if it is not involved in the rule application and kept in at least one of the premise(es); *weak* if it is not involved in the rule application and does not occur in any of the premise(es). Occurrences of formulas which are neither side nor weak are called *principal*. For example, in $(\vee^\circ)$, the explicit $A \vee B^\circ$ is principal and formulas in Σ and $G\{\ \}$ are all side. For the only structural rule (lift), we call all the formula occurrences by side formulas.

The following proposition is shown by induction on the complexity of A.

Proposition 2.6. *Generalized initial sequents of the form* $G\{\Sigma, A^\bullet, A^\circ\}$ *are derivable in* $\mathbf{C}_{\mathsf{B}^+}$.

2.1 Semantic interpretation and soundness

In what follows, we establish the soundness of $\mathbf{C}_{\mathsf{B}^+}$. To this end, the satisfaction relation introduced in the previous section is extended from formulas to nested sequents, in accordance with the interpretation outlined in Remark 2.4.

Definition 2.7 (Semantic interpretation)**.** Let Σ be a sequent and $\mathscr{M} = (\mathscr{O}, K, R, V)$ be a B^+-model and $x, y, z \in K$.

For a left-sided sequent $\Sigma = A_1^\bullet, \ldots, A_m^\bullet, \langle \Phi_{11} \mathbin{/\!/} \Phi_{12} \rangle, \ldots, \langle \Phi_{j1} \mathbin{/\!/} \Phi_{j2} \rangle$, we define $x \Vdash \Sigma$ iff

- for any $i \in \{1, \ldots, m\}$, $x \Vdash A_i^\bullet$; and
- for any $j \in \{1, \ldots, n\}$, there exist y, z with $Rxyz$, $y \Vdash \Phi_{j1}$ and $z \Vdash \Phi_{j2}$

For a full sequent, we define the interpretation according to the location of the output formula:

- $x \Vdash \Sigma, A^\circ$ iff $x \Vdash \Sigma$ implies $x \Vdash A$
- $x \Vdash \Sigma, \langle \Phi_1 \mathbin{/\!/} \Phi_2 \rangle$ (where Φ_1 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $z \Vdash \Phi_2$ implies $y \Vdash \Phi_1$
- $x \Vdash \Sigma, \langle \Phi_1 \mathbin{/\!/} \Phi_2 \rangle$ (where Φ_2 is full) iff if $x \Vdash \Sigma$ then for any y, z with $Rxyz$, $y \Vdash \Phi_1$ implies $z \Vdash \Phi_2$

We say a sequent Σ is B^+-satisfiable if there is a B^+-model $\mathscr{M}$ and a regular world w in it such that $\mathscr{M}, w \Vdash \Sigma$; and Σ is B^+-*valid* if $\mathscr{M}, w \Vdash \Sigma$ for any B^+-model $\mathscr{M}$ and any regular world w in it.

Intuitively, the satisfaction of a full sequent says if the left-sided part is satisfied, so is the unique output formula. Notice that we interpret a block in different ways according to whether it contains the output formula and where the exact location of the output formula is. Similar practice for interpreting sequents with single output is also used in the context of intuitionistic modal logic [34]. According to the definition, for a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$, if $\mathscr{M}, w \nVdash G\{\Sigma, A^\circ\}$, then there exists some $x \in \mathscr{K}$ such that $x \nVdash \Sigma, A^\circ$.

To show the soundness of the calculus, we first verify each rule of $\mathbf{C}_{\mathsf{B}^+}$ is valid. For a rule (r) in $\mathbf{C}_{\mathsf{B}^+}$, we say (r) is *valid* if whenever all of its premise are valid, the conclusion is also valid.

Lemma 2.8. All the rules in $\mathbf{C}_{\mathsf{B}^+}$ are B^+-valid.

Proof. We prove the basic case when $G\{\ \}$ is empty, the general case is then proved by induction on the structure of the context. We only show the two rules for implication as other cases are straightforward.

$\to^\bullet$: Let $\Sigma = \Sigma', A \to B^\bullet$ and the output formula being in Φ_1. Assume the premises are valid but the conclusion is not valid. Since $\nVdash \Sigma', A \to B^\bullet, \langle \Phi_1 \mathbin{/\!/} \Phi_2 \rangle$, there exists a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $x \in \mathscr{O}$ such that $\mathscr{M}, x \nVdash \Sigma', A \to B^\bullet, \langle \Phi_1 \mathbin{/\!/} \Phi_2 \rangle$. This implies $\mathscr{M}, x \Vdash \Sigma'$. If Φ_1 is full, then there exist $y, z \in K$ such that $Rxyz$, $z \Vdash \Phi_2$ and $y \nVdash \Phi_1$. Since $x \Vdash A \to B$, by the semantic clause of implication we have that for all u', v' with $Rxu'v'$, if $u' \Vdash A$ then $v' \Vdash B$. Hence for u', v' we obtain $u' \nVdash A$ or $v' \Vdash B$. If $u' \nVdash A$, then $\mathscr{M}, x \nVdash \Sigma, \langle \Phi_1^*, A^\circ \mathbin{/\!/} \Phi_2^* \rangle$, contradicting the validity of the first premise. If $v' \Vdash B$, then $\mathscr{M}, x \nVdash \Sigma, \langle \Phi_1 \mathbin{/\!/} \Phi_2, B^\bullet \rangle$, contradicting the validity of the second premise.

$\to^\circ$: Assume the premise is valid but the conclusion is not valid. Since $\nVdash \Sigma, A \to B^\circ$, it implies there exists a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $x \in \mathscr{O}$ such that $\mathscr{M}, x \nVdash \Sigma, A \to B^\circ$, which further implies $\mathscr{M}, x \Vdash \Sigma$ and $\mathscr{M}, x \nVdash A \to B$. Hence there exist $y, z \in K$ such that $Rxyz$, $y \Vdash A$, and $z \nVdash B$. Meanwhile, since $\Vdash \Sigma, \langle A^\bullet \mathbin{/\!/} B^\circ \rangle$, we have $\mathscr{M}, x \Vdash \langle A^\bullet \mathbin{/\!/} B^\circ \rangle$. Since the forcing of $\langle A^\bullet \mathbin{/\!/} B^\circ \rangle$ at x requires that for all y', z' with $Rxy'z'$, if $y' \Vdash A$ then $z' \Vdash B$, so by $y \Vdash A$ we have $z \Vdash B$, a contradiction.

This completes the proof. □

The soundness of $\mathbf{C}_{\mathsf{B}^+\mathsf{K}\square}$ then follows as usual by induction on the structure of a derivation.

Theorem 2.9 (Soundness of $\mathbf{C}_{\mathsf{B}^+}$)**.** *Let Σ be a sequent. If Σ is derivable in $\mathbf{C}_{\mathsf{B}^+}$, then it is B^+-valid.*

2.2 Completeness and (cut)-admissibility

In this section, our aim is twofold. First, we establish the completeness of the calculus with cut rule by deriving the axioms of the Hilbert system $\mathscr{H}_{\mathsf{B}^+}$ within $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$. On the basis of these results, and by proving some preliminary structural properties of $\mathbf{C}_{\mathsf{B}^+}$ together with the invertibility of certain rules, we then show that (cut) is an admissible rule. As a consequence, the calculus $\mathbf{C}_{\mathsf{B}^+}$ itself is complete.

Definition 2.10 (Cut rule)**.** The nested (cut)-rule is defined as

$$\frac{G^*\{\Sigma^*, A^\circ\} \qquad G\{\Sigma, A^\bullet\}}{G\{\Sigma\}}\ (\mathsf{cut})$$

Theorem 2.11 (Completeness of $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$)**.** *All the axioms and rules in $\mathscr{H}_{\mathsf{B}^+}$ are derivable in $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$. Consequently, if a formula $A \in \mathsf{Frm}$ is valid, then it is provable in $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$.*

Proof. The derivations of the axioms and rules are found in the Appendix. □

The rest of this section is devoted to the proof of cut-admissibility. To this end, we begin by introducing the following preliminary notions.

Definition 2.12 ((hp-)admissibility and (hp-)invertibility)**.** A rule $\mathscr{R}$ is called *admissible* in a calculus if, whenever all of its premises are derivable at height n, its conclusion is also derivable with height at most m. If $m \leq n$, then $\mathscr{R}$ is called *height-preserving admissible* (*hp-admissible* for short). A rule $\mathscr{R}$ is called *height-preserving invertible* (*hp-invertible* for short) in a calculus if, whenever its conclusion is derivable with height at most m, all of its premises are derivable with height at most m as well.

Proposition 2.13. *The rules $\wedge^\bullet, \vee^\bullet$ and (lift) are hp-invertible in $\mathbf{C}_{\mathsf{B}^+}$.*

Proposition 2.14. *The following structural rules are hp-admissible in $\mathbf{C}_{\mathsf{B}^+}$:*

$$\frac{G\{A^\bullet, A^\bullet, \Sigma\}}{G\{A^\bullet, \Sigma\}}\ \mathsf{c} \qquad \frac{G\{\Sigma\}}{G\{\Lambda, \Sigma\}}\ \mathsf{w} \qquad \frac{G\{\Sigma, \langle \Phi_1 \mathbin{/\!/} \Phi_2\rangle, \langle \Psi_1 \mathbin{/\!/} \Psi_2\rangle\}}{G\{\Sigma, \langle \Phi_1, \Psi_1 \mathbin{/\!/} \Psi_2, \Phi_2\rangle\}}\ \mathsf{m}$$

Proof. The proof follows from a routine induction on the height of the derivation. □

Further structural rules are considered for establishing the completeness of $\mathbf{C}_{\mathsf{B}^+}$, but they need not be taken as primitive. We consider the following two rules:

$$\frac{\langle \Phi_1 \mathbin{/\!/} \Phi_2\rangle}{G\{\Phi_1, \Phi_2\}}\ \mathsf{R1} \qquad \frac{\Sigma, \langle \Phi_1, \langle \Psi_1 \mathbin{/\!/} \Psi_2\rangle \mathbin{/\!/} \Phi_2\rangle}{\Sigma, \langle \Phi_1 \mathbin{/\!/} \Phi_2, \langle \Psi_1 \mathbin{/\!/} \Psi_2\rangle\rangle}\ \mathsf{R2}$$

Rule R1 reflects frame condition (F1), expressing the reflexivity of $\leq$ by allowing any nested component to be treated as a whole relevant block. Rule R2 corresponds to frame condition (F2) and allows a relevant block occurring in the second component of a top-level block to be moved to the first component. Moreover, the semantics given in Definition 2.7 aligns well with these rules, ensuring their validity in full generality and making any restriction on the position of the output formula unnecessary.

We show both R1 and R2 are 'correct' structural rules in $\mathbf{C}_{\mathsf{B}^+}$ in the sense that they are semantically valid and syntactically hp-admissible in the calculus.

Proposition 2.15. *The rules* R1 *and* R2 *are* B^+*-valid.*

Proof. First, we prove the validity of R1. Since $\langle \Phi_1 /\!/ \Phi_2 \rangle$ is full, the output formula occurs either in Φ_1 or Φ_2. We only show the former case as the latter is similar. Assume the premise is valid but the conclusion is not. Then there exist a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $a \in K$ such that $a \Vdash \Phi_2$ and $a \nVdash \Phi_1$. Since $a \leq a$, by (F1) there exists $x \in \mathscr{O}$ such that $Rxaa$. By the semantic clause for $\langle \Phi_1 /\!/ \Phi_2 \rangle$, $x \Vdash \langle \Phi_1 /\!/ \Phi_2 \rangle$ requires that for all $y, z \in K$, if $Rxyz$ and $z \Vdash \Phi_2$ then $y \Vdash \Phi_1$. Taking $y = a$ and $z = a$, since $Rxaa$, $a \Vdash \Phi_2$ and $a \nVdash \Phi_1$, we obtain $x \nVdash \langle \Phi_1 /\!/ \Phi_2 \rangle$, contradicting the validity of the premise.

Next, we move towards the validity of R2. We only show two cases when the output formula occurs in Σ and Φ_1 as the other cases are similar.

- The output formula occurs in Σ. Assume the conclusion is not valid, this means there exists a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$ such that $w \nVdash \Sigma$ and $w \Vdash \Sigma^*, \langle \Phi_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle$. By definition, we have $w \Vdash \langle \Phi_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle$, which is a left-sided sequent. This further implies there exist $a, b, c, d \in K$ such that $Rwab, a \Vdash \Phi_1, b \Vdash \Phi_2$ and $Rbcd, c \Vdash \Psi_1$ and $d \Vdash \Psi_2$. Since the premise is valid, we have $w \Vdash \Sigma, \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle$. Since $w \nVdash \Sigma$, we have $w \nVdash \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle$. This implies that for any a', b', c', d' with $Rwa'b'$ and $Ra'c'd'$, either $a' \nVdash \Phi_1$ or $b' \nVdash \Phi_2$ or $c' \nVdash \Psi_1$ or $d' \nVdash \Psi_2$. Taking a, b, c, d given by the invalidity of the conclusion, we have $a \Vdash \Phi_1$ and $b \Vdash \Phi_2$; moreover, from $Rbcd$ and $Rwab$, by (F2) we have $Racd$, hence $c \nVdash \Psi_1$ or $d \nVdash \Psi_2$. Since $c \Vdash \Psi_1$ and $d \Vdash \Psi_2$, a contradiction.
- The output formula occurs in Φ_1. Assume the conclusion is not valid, this means there exists a model $\mathscr{M} = (\mathscr{O}, K, R, V)$ and $w \in \mathscr{O}$ such that $w \Vdash \Sigma$ and $w \nVdash \langle \Phi_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle$. By Definition 2.7 there exist $a, b \in K$ such that $Rwab, a \nVdash \Phi_1$ and $b \Vdash \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle$. Since $\langle \Psi_1 /\!/ \Psi_2 \rangle$ is left-sided, this further implies for any $c, d \in K$ with $Rbcd$ it holds $c \Vdash \Psi_1$ and $d \Vdash \Psi_2$. Since the premise is valid we have $w \Vdash \Sigma, \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle$. Since $b \Vdash \Phi_2$, we have $a \Vdash \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle$. Recall that $a \nVdash \Phi_1$ and $\langle \Psi_1 /\!/ \Psi_2 \rangle$ is left-sided, hence $a \nVdash \langle \Psi_1 /\!/ \Psi_2 \rangle$. This implies that for any c', d' with $Rac'd'$, either $c' \nVdash \Psi_1$ or $d' \nVdash \Psi_2$. Taking c, d obtained above, from $Rbcd$ and $Rwab$ we obtain $Racd$ by (F2), since $c \Vdash \Psi_1$ and $d \Vdash \Psi_2$, a contradiction.

This completes the proof. □

Proposition 2.16. *The rules* R1 *and* R2 *are hp-admissible in* $\mathbf{C}_{\mathsf{B}^+}$.

Proof. First, we provide a detailed proof of the admissibility of R2. The proof proceeds by induction on the height n of the derivation. If $n = 0$, the conclusion follows straightforwardly. If $n > 0$, let us consider the last rule r applied in the derivation of the premise.

Case 1. r is applied in Σ. If r is one of the propositional rules, the case is straightforward: by considering a derivation of the premise ending with an application of r, the induction hypothesis yields a derivation of the desired conclusion, to which r can then be applied again. As an example, take $\Sigma = \Sigma', A \to B^\bullet$ and $r = \to^\bullet$ is applied to $A \to B^\bullet$ and $\langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Sigma', A \to B^\bullet, \langle \Phi_1^*, A^\circ, \langle \Psi_1^* /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle} \quad \dfrac{\vdots}{\Sigma', A \to B^\bullet, \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2, B^\bullet \rangle}}{\Sigma', A \to B^\bullet, \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle}\, r}{\Sigma', A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{R2} \quad \rightsquigarrow
$$

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Sigma', A \to B^\bullet, \langle \Phi_1^*, A^\circ, \langle \Psi_1^* /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle}}{\Sigma', A \to B^\bullet, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^*, \langle \Psi_1^* /\!/ \Psi_2^* \rangle \rangle}\, \mathsf{IH} \quad \dfrac{\dfrac{\vdots}{\Sigma', A \to B^\bullet, \langle \Phi_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2, B^\bullet \rangle}}{\Sigma', A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2, B^\bullet, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{IH}}{\Sigma', A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, r
$$

Case 2. r is applied in Φ_1.

- If r is a rule for either $\wedge$ or $\vee$, no difficulties arise.
- Let $\Phi_1 = \Phi'_1, A \to B^\bullet$ and $r = \to^\bullet$ is applied to $A \to B^\bullet$ and $\langle \Psi_1 /\!/ \Psi_2 \rangle$. We have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet, \langle \Psi_1^*, A^\circ /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Psi_1 /\!/ \Psi_2, B^\bullet \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle}\, r}{\Sigma, \langle \Phi'_1, A \to B^\bullet /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{R2} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet, \langle \Psi_1^*, A^\circ /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle\end{array}}{\Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet /\!/ \Phi_2^*, \langle \Psi_1^*, A^\circ /\!/ \Psi_2^* \rangle \rangle}\, \mathrm{IH}}{\Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet /\!/ \Phi_2^*, A \to B^\bullet, \langle \Psi_1^*, A^\circ /\!/ \Psi_2^* \rangle \rangle}\, \mathsf{w} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Psi_1 /\!/ \Psi_2, B^\bullet \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\bullet /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2, B^\bullet \rangle \rangle}\, \mathrm{IH}}{\Sigma, \langle \Phi'_1, A \to B^\bullet /\!/ \Phi_2, A \to B^\bullet, \langle \Psi_1 /\!/ \Psi_2, B^\bullet \rangle \rangle}\, \mathsf{w}}{\Sigma, \langle \Phi'_1, A \to B^\bullet /\!/ \Phi_2, A \to B^\bullet, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, r}{\Sigma, \langle \Phi'_1, A \to B^\bullet /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \text{hp-inv. (lift)}$$

- Let $\Phi_1 = \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2 \rangle$ and $r = \to^\bullet$ is applied to $A \to B^\bullet$ and $\langle \Theta_1 /\!/ \Theta_2 \rangle$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet, \langle \Theta_1^*, A^\circ /\!/ \Theta_2^* \rangle, \langle \Psi_1^* /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2, B^\bullet \rangle, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2 \rangle, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle}\, r}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2 \rangle /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{R2} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma^*, \langle {\Phi'_1}^*, A \to B^\bullet, \langle \Theta_1^*, A^\circ /\!/ \Theta_2^* \rangle, \langle \Psi_1^* /\!/ \Psi_2^* \rangle /\!/ \Phi_2^* \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1^*, A^\circ /\!/ \Theta_2^* \rangle /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathrm{IH} \qquad \dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2, B^\bullet \rangle, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2, B^\bullet \rangle /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathrm{IH}}{\Sigma, \langle \Phi'_1, A \to B^\bullet, \langle \Theta_1 /\!/ \Theta_2 \rangle /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, r$$

  Other nested cases happened within Φ_1 are similar.

- Let $\Phi_1 = \Phi'_1, A \to B^\circ$ and $r = \to^\circ$. We consider the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, \langle A^\bullet /\!/ B^\circ \rangle, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, A \to B^\circ, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle}\, r}{\Sigma, \langle \Phi'_1, A \to B^\circ /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{R2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle \Phi'_1, \langle A^\bullet /\!/ B^\circ \rangle, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle\end{array}}{\Sigma, \langle \Phi'_1, \langle A^\bullet /\!/ B^\circ \rangle /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathrm{IH}}{\Sigma, \langle \Phi'_1, A \to B^\circ /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, r$$

- Let $\Phi_1 = A^\bullet, \Phi'_1$, with $A^\bullet$ being involved in $r = (\mathsf{lift})$. We consider the following derivation.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle A^\bullet, \Phi'_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ A^\bullet, \Phi_2 \rangle\end{array}}{\Sigma, \langle A^\bullet, \Phi'_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ \Phi_2 \rangle}\, r}{\Sigma, \langle A^\bullet, \Phi'_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathsf{R2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma, \langle A^\bullet, \Phi'_1, \langle \Psi_1 /\!/ \Psi_2 \rangle /\!/ A^\bullet, \Phi_2 \rangle\end{array}}{\Sigma, \langle A^\bullet, \Phi'_1 /\!/ A^\bullet, \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, \mathrm{IH}}{\Sigma, \langle A^\bullet, \Phi'_1 /\!/ \Phi_2, \langle \Psi_1 /\!/ \Psi_2 \rangle \rangle}\, r$$

Case 3. r is applied in $\langle \Psi_1 /\!/ \Psi_2 \rangle$. We distinguish two subcases, depending on whether r is applied in Ψ_1 or in Ψ_2. These cases present no particular difficulties. By examining the premises of r, we apply the induction hypothesis to each of them in order to relocate the relevant block $\langle \Psi_1 /\!/ \Psi_2 \rangle$ into the second component. The desired conclusion is then obtained by reapplying r.

Case 4. r is applied in Φ_2. Similarly to **Case 2**.

Now, we turn to R1. The proof proceeds in a similar and even simpler way. We only show the case when the premise $\langle \Phi_1 /\!/ \Phi_2 \rangle$ is derived from (lift). In this case, assume the rule is applied to some $A^\bullet \in \Phi_1$ and we further denote $\Phi_1 = A^\bullet, \Phi'_1$. Then we have the following transformation of the derivation:

$$\dfrac{\dfrac{\overset{\vdots}{\langle A^\bullet,\Phi'_1 /\!/ A^\bullet,\Phi_2\rangle}}{\langle A^\bullet,\Phi'_1 /\!/ \Phi_2\rangle}\,(\mathsf{lift})}{G\{A^\bullet,\Phi'_1,\Phi_2\}}\,\mathsf{R1} \quad\rightsquigarrow\quad \dfrac{\dfrac{\overset{\vdots}{\langle A^\bullet,\Phi'_1 /\!/ A^\bullet,\Phi_2\rangle}}{G\{A^\bullet,A^\bullet,\Phi'_1,\Phi_2\}}\,\mathrm{IH}}{G\{A^\bullet,\Phi'_1,\Phi_2\}}\,\mathsf{c}$$

This completes the proof. □

Theorem 2.17 (Cut-admissibility)**.** (cut) *is admissible in* $\mathbf{C}_{\mathsf{B}^+}$.

Proof. Let us consider an application of (cut) of the following form in a derivation $\mathscr{D}$:

$$\dfrac{\overset{\mathscr{D}_1}{G\{\Sigma^*,A^\circ\}} \qquad \overset{\mathscr{D}_2}{G\{A^\bullet,\Sigma\}}}{G\{\Sigma\}}\,(\mathsf{cut})$$

We define as usual the rank of each application of (cut) as an ordered pair $(|A|,h_1+h_2)$ where $|A|$ denotes the size of the cut formula A and h_1,h_2 denote the height of the two premises of (cut) in $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. We prove by induction on the rank of (cut).

For the basic step, suppose $A=p$. If the left premise $G\{\Sigma^*,A^\circ\}$ is an initial sequent, then $A^\bullet\in\widetilde{\Sigma^*}$. Applying contraction to the right premise, we obtain the conclusion from the premise directly. If the right premise $G\{\Sigma,A^\bullet\}$ is an initial sequent, then either $A^\circ\in\widetilde{\Sigma}$ or some other nested component containing the output formula is initial. For the former, it implies $\Sigma^*,p^\circ=\Sigma$, so we just take the left premise as it is identical to the conclusion; for the latter, it is easy to see by removing $A^\bullet$ from the right premise, the sequent which remains $G\{\Sigma\}$ is still initial, which is exactly the conclusion of (cut).

For the inductive step, let us further assume the last rule applied in $\mathscr{D}_1$ and $\mathscr{D}_2$ are r_1 and r_2 respectively. We proceed by discussing different roles of A° and $A^\bullet$ in r_1 and r_2. Note that no rules in the calculus introduce weak formulas in the conclusion, so we only need to consider whether the cut formulas are principal or side. Thus we have the following cases: **Case 1.** A° is principal in r_1 and $A^\bullet$ is principal in r_2; **Case 2.** A° is side in r_1; **Case 3.** $A^\bullet$ is side in r_2 and A° is principal in r_1. We demonstrate each case in turn.

Case 1. A° is principal in r_1 and $A^\bullet$ is principal in r_2. Then we have the following sub-cases according to the shape of A:

- $A=B\wedge C$. Then we have the following transformation of the derivation

$$\dfrac{\dfrac{\overset{\vdots}{G^*\{\Sigma^*,B^\circ\}} \quad \overset{\vdots}{G^*\{\Sigma^*,C^\circ\}}}{G^*\{\Sigma^*,B\wedge C^\circ\}}\,r_1=\wedge^\circ \quad \dfrac{\overset{\vdots}{G\{\Sigma,B^\bullet,C^\bullet\}}}{G\{\Sigma,B\wedge C^\bullet\}}\,r_2=\wedge^\bullet}{G\{\Sigma\}} \quad\rightsquigarrow\quad \dfrac{\overset{\vdots}{G^*\{\Sigma^*,C^\circ\}} \quad \dfrac{\dfrac{\overset{\vdots}{G^*\{\Sigma^*,B^\circ\}}}{G^*\{\Sigma^*,B^\circ,C^\bullet\}}\,\mathsf{w} \quad \overset{\vdots}{G\{\Sigma,B^\bullet,C^\bullet\}}}{G\{\Sigma,C^\bullet\}}\,\mathrm{IH}}{G\{\Sigma\}}\,\mathrm{IH}$$

- $A=B\vee C$. Assume w.l.o.g the left premise $G^*\{\Sigma^*,B\vee C^\circ\}$ is derived from $G^*\{\Sigma^*,B^\circ\}$. Then we have the following transformation of the derivation

$$\dfrac{\dfrac{\overset{\vdots}{G^*\{\Sigma^*,B^\circ\}}}{G^*\{\Sigma^*,B\vee C^\circ\}}\,r_1=\vee^\circ \quad \dfrac{\overset{\vdots}{G\{\Sigma,B^\bullet\}} \quad \overset{\vdots}{G\{\Sigma,C^\bullet\}}}{G\{\Sigma,B\vee C^\bullet\}}\,r_2=\vee^\bullet}{G\{\Sigma\}} \quad\rightsquigarrow\quad \dfrac{\overset{\vdots}{G^*\{\Sigma^*,B^\circ\}} \quad \overset{\vdots}{G\{\Sigma,B^\bullet\}}}{G\{\Sigma\}}\,\mathrm{IH}$$

- $A = B \to C$. Assume $\Sigma = \Sigma', \langle \Phi_1 /\!/ \Phi_2 \rangle$. Then we have the following transformation of the derivation

$$\dfrac{\dfrac{\overset{\vdots}{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*\rangle,\langle B^\bullet /\!/ C^\circ\rangle\}}}{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*\rangle,B\to C^\circ\}}\,r_1={\to}^\circ \qquad \dfrac{\overset{\vdots}{G^*\{\Sigma'^*,\langle\Phi_1^*,B^\circ /\!/ \Phi_2^*\rangle,B\to C^\bullet\}} \quad \overset{\vdots}{G\{\Sigma,\langle\Phi_1 /\!/ \Phi_2,C^\bullet\rangle,B\to C^\bullet\}}}{G\{\Sigma',\langle\Phi_1 /\!/ \Phi_2\rangle,B\to C^\bullet\}}\,r_2={\to}^\bullet}{G\{\Sigma',\langle\Phi_1 /\!/ \Phi_2\rangle\}}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*\rangle,B\to C^\circ\} \quad \overset{\vdots}{G^*\{\Sigma'^*,\langle\Phi_1^*,B^\circ /\!/ \Phi_2^*\rangle,B\to C^\bullet\}}}{G^*\{\Sigma'^*,\langle\Phi_1^*,B^\circ /\!/ \Phi_2^*\rangle\}}\,\text{IH} \quad \dfrac{\overset{\vdots}{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*\rangle,\langle B^\bullet /\!/ C^\circ\rangle\}}}{G^*\{\Sigma'^*,\langle\Phi_1^*,B^\bullet /\!/ \Phi_2^*,C^\circ\rangle\}}\,\mathsf{m}}{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*,C^\circ\rangle\}}\,\text{IH} \quad \dfrac{G^*\{\Sigma'^*,\langle\Phi_1^* /\!/ \Phi_2^*\rangle,B\to C^\circ\} \quad \overset{\vdots}{G\{\Sigma,\langle\Phi_1 /\!/ \Phi_2,C^\bullet\rangle,B\to C^\bullet\}}}{G\{\Sigma,\langle\Phi_1 /\!/ \Phi_2,C^\bullet\rangle\}}\,\text{IH}}{G\{\Sigma,\langle\Phi_1 /\!/ \Phi_2\rangle\}}\,\text{IH}$$

Case 2. A° is side in r_1. Let us discuss the possibilities of r_1. Note that in this case r_1 cannot be a $\circ$-rule.

- $r_1 \in \{\vee^\bullet, \wedge^\bullet, (\mathsf{lift})\}$. Then by hp-invertibility of the rule, we can simulate the same rule application and apply (cut) for the premise of r_1 and inverted conclusion of r_2. For example, assume $\Sigma^* = \Sigma'^*, B \wedge C^\bullet$ and r_1 is applied to $B \wedge C^\bullet$. Then we have the following transformation of the derivation

$$\dfrac{\dfrac{\overset{\vdots}{G\{\Sigma'^*,B^\bullet,C^\bullet,A^\circ\}}}{G\{\Sigma'^*,B\wedge C^\bullet,A^\circ\}}\,r_1=\wedge^\bullet \qquad \overset{\vdots}{G\{\Sigma',B\wedge C^\bullet,A^\bullet\}}}{G\{\Sigma',B\wedge C^\bullet\}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{G\{\Sigma'^*,B^\bullet,C^\bullet,A^\circ\}} \qquad \dfrac{\overset{\vdots}{G\{\Sigma',B\wedge C^\bullet,A^\bullet\}}}{G\{\Sigma',B^\bullet,C^\bullet,A^\bullet\}}\,\text{hp-inver}}{G\{\Sigma',B^\bullet,C^\bullet\}}}{G\{\Sigma',B\wedge C^\bullet\}}$$

  Other cases when r_1 is applied in nested component contained in $G\{\ \}$ are similar.

- $r_1 ={\to}^\bullet$ is applied to some $B \to C^\bullet$. We can transform the derivation by switching the order of (cut) and r_1. For example, assume w.l.o.g. $G^*\{\Sigma^*, A^\circ\} = {\Sigma_0}^*, B \to C^\bullet, \langle \Phi_1^*, /\!/ \Phi_2^*, \langle \Psi^* /\!/ \Sigma^*, A^\circ \rangle\rangle$ (other cases are treated similarly),

$$\dfrac{\dfrac{\overset{\vdots}{{\Sigma_0}^*,B\to C^\bullet,\langle\Phi_1^*,B^\circ,/\!/\Phi_2^*,\langle\Psi^* /\!/ \Sigma^*\rangle\rangle} \quad \overset{\vdots}{{\Sigma_0}^*,B\to C^\bullet,\langle\Phi_1^* /\!/ \Phi_2^*,C^\bullet,\langle\Psi^* /\!/ \Sigma^*,A^\circ\rangle\rangle}}{{\Sigma_0}^*,B\to C^\bullet,\langle\Phi_1^* /\!/ \Phi_2^*,\langle\Psi^* /\!/ \Sigma^*,A^\circ\rangle\rangle} \qquad \overset{\vdots}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma,A^\bullet\rangle\rangle}}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma\rangle\rangle}$$

$$\rightsquigarrow$$

$$\dfrac{\overset{\vdots}{{\Sigma_0}^*,B\to C^\bullet,\langle\Phi_1^*,B^\circ,/\!/\Phi_2^*,\langle\Psi^* /\!/ \Sigma^*\rangle\rangle} \qquad \dfrac{\overset{\vdots}{{\Sigma_0}^*,B\to C^\bullet,\langle\Phi_1^* /\!/ \Phi_2^*,C^\bullet,\langle\Psi^* /\!/ \Sigma^*,A^\circ\rangle\rangle} \quad \dfrac{\overset{\vdots}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma,A^\bullet\rangle\rangle}}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma,A^\bullet\rangle,C^\bullet\rangle}\,\mathsf{w}}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma\rangle,C^\bullet\rangle}\,\text{IH}}{\Sigma_0,B\to C^\bullet,\langle\Phi_1 /\!/ \Phi_2,\langle\Psi /\!/ \Sigma\rangle\rangle}\,r_2={\to}^\bullet$$

Case 3. $A^\bullet$ is side in r_2 and A° is principal in r_1.

In this case, let us discuss different shapes of A. If $A = B \vee C$ or $B \wedge C$, then it follows directly by invertibility of $A^\bullet$ in the right premise of (cut). Thus we only need to consider the case when $A = B \to C$. We further discuss the possibilities of r_2.

- $r_2 \in \{\vee^\bullet, \wedge^\bullet\}$ or $r_2 = (\mathsf{lift})$ but not applied to $A^\bullet$, dual to the first case in **Case 2**.
- $r_2 \in \{\vee^\circ, \wedge^\circ, {\to}^\circ\}$. In these cases, we just permute the rule order of r_2 and (cut).
- $r_2 = (\mathsf{lift})$ and applied to $A^\bullet$. In this case we can further assume $G\{\Sigma, A^\bullet\} = \Sigma_0, \langle A^\bullet, \Phi_1 /\!/ \Phi_2 \rangle$ where $G\{\cdot\} = \Sigma_0, \langle \cdot /\!/ \Phi_2 \rangle$ and $\Sigma = \Phi_1$. Then we have the derivation $\mathscr{D}_2$ is unfolded as

$$\dfrac{\overset{\mathscr{D}_2'}{\Sigma_0,\langle A^\bullet,\Phi_1 /\!/ A^\bullet,\Phi_2\rangle}}{\Sigma_0,\langle A^\bullet,\Phi_1 /\!/ \Phi_2\rangle}\,r_2=(\mathsf{lift})$$

Let us further consider the rule above r_2, namely the last rule applied in $\mathscr{D}'_2$, and call it by r_3. If r_3 is a $\circ$-rule or another (lift) applied to a formula occurrence rather than $A^\bullet$, we only need to permute the order of r_2 and r_3. For example, assume $r_3 = \rightarrow^\circ$ and is applied to $D \rightarrow E^\circ \in \Phi_2$. Let us write $\Phi_2 = \Phi_2^*, D \rightarrow E^\circ$, then we transform the derivation as

$$\dfrac{\overset{\vdots}{\Sigma_0^*, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle} \quad \dfrac{\dfrac{\overset{\vdots}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2^*, \langle D^\bullet /\!/ E^\circ \rangle\rangle}}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2^*, D \rightarrow E^\circ \rangle}\; r_3 = \rightarrow^\circ}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ \Phi_2^*, D \rightarrow E^\circ \rangle}\; r_2 = (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ \Phi_2^*, D \rightarrow E^\circ \rangle} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\vdots}{\Sigma_0^*, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle}}{\Sigma_0^*, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^*, \langle D^\bullet /\!/ - \rangle\rangle}\; \mathsf{w} \quad \dfrac{\overset{\vdots}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2^*, \langle D^\bullet /\!/ E^\circ \rangle\rangle}}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ \Phi_2^*, \langle D^\bullet /\!/ E^\circ \rangle\rangle}\; r_2 = (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ \Phi_2^*, \langle D^\bullet /\!/ E^\circ \rangle\rangle}\; \mathrm{IH}}{\Sigma_0, \langle \Phi_1 /\!/ \Phi_2^*, D \rightarrow E^\circ \rangle}\; r_3 = \rightarrow^\circ$$

If $r_3 = (\mathsf{lift})$ and is applied to $A^\bullet$, then we have the following transformation relying on the hp-admissibility of contraction (cf. Proposition 2.14):

$$\dfrac{\overset{\vdots}{\Sigma_0^*, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle} \quad \dfrac{\dfrac{\overset{\vdots}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, A^\bullet, \Phi_2 \rangle}}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2 \rangle}\; r_3 = (\mathsf{lift})}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ \Phi_2 \rangle}\; r_2 = (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ \Phi_2 \rangle} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Sigma_0^*, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle} \quad \dfrac{\dfrac{\overset{\vdots}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, A^\bullet, \Phi_2 \rangle}}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2 \rangle}\; \mathsf{c}}{\Sigma_0, \langle A^\bullet, \Phi_1 /\!/ \Phi_2 \rangle}\; r_2 = (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ \Phi_2 \rangle}\; \mathrm{IH}$$

Lastly, the only tricky case remains is when $r_3 = \rightarrow^\bullet$ and $A^\bullet$ in the same block of Φ_2 is principal. Let $\Phi_2 = \Phi'_2, \langle \Psi_1 /\!/ \Psi_2 \rangle$, then we transform the derivation as

$$\dfrac{\dfrac{\overset{\vdots}{\Sigma_0^*, \langle \Phi_1^*, \langle B^\bullet /\!/ C^\circ \rangle /\!/ {\Phi'_2}^*, \langle \Psi_1^* /\!/ \Psi_2^* \rangle\rangle}}{\Sigma_0^*, \langle \Phi_1^*, B \rightarrow C^\circ /\!/ {\Phi'_2}^*, \langle \Psi_1^* /\!/ \Psi_2^* \rangle\rangle}\; r_1 \quad \dfrac{\dfrac{\overset{\vdots}{\Sigma_0^*, \langle B \rightarrow C^\bullet, \Phi_1^* /\!/ B \rightarrow C^\bullet, {\Phi'_2}^*, \langle \Psi_1^*, B^\circ /\!/ \Psi_2^* \rangle\rangle} \quad \overset{\vdots}{\Sigma_0, \langle B \rightarrow C^\bullet, \Phi_1 /\!/ B \rightarrow C^\bullet, \Phi'_2, \langle \Psi_1 /\!/ \Psi_2, C^\bullet \rangle\rangle}}{\Sigma_0, \langle B \rightarrow C^\bullet, \Phi_1 /\!/ B \rightarrow C^\bullet, \Phi'_2, \langle \Psi_1 /\!/ \Psi_2 \rangle\rangle}\; r_3 = \rightarrow^\bullet}{\Sigma_0, \langle B \rightarrow C^\bullet, \Phi_1 /\!/ \Phi'_2, \langle \Psi_1 /\!/ \Psi_2 \rangle\rangle}\; r_2 = (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ {\Phi'_2}^*, \langle \Psi_1 /\!/ \Psi_2 \rangle\rangle}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Sigma_0{}^*, \langle \Phi_1^*, B \rightarrow C^\circ /\!/ {\Phi'_2}^*, \langle \Psi_1 /\!/ \Psi_2 \rangle\rangle \quad \dfrac{\overset{\vdots}{\Sigma_0^*, \langle B \rightarrow C^\bullet, \Phi_1^* /\!/ B \rightarrow C^\bullet, {\Phi'_2}^*, \langle \Psi_1^*, B^\circ /\!/ \Psi_2^* \rangle\rangle}}{\Sigma_0^*, \langle B \rightarrow C^\bullet, \Phi_1^* /\!/ {\Phi'_2}^*, \langle \Psi_1^*, B^\circ /\!/ \Psi_2^* \rangle\rangle}\; (\mathsf{lift})}{\Sigma_0^*, \langle \Phi_1^* /\!/ {\Phi'_2}^*, \langle \Psi_1^*, B^\circ /\!/ \Psi_2^* \rangle\rangle}\; \mathrm{IH} \quad \dfrac{\dfrac{\Sigma_0{}^*, \langle \Phi_1^*, \langle B^\bullet /\!/ C^\circ \rangle /\!/ {\Phi'_2}^*, \langle \Psi_1^* /\!/ \Psi_2^* \rangle\rangle}{\Sigma_0{}^*, \langle \Phi_1^* /\!/ {\Phi'_2}^*, \langle B^\bullet /\!/ C^\circ \rangle, \langle \Psi_1^* /\!/ \Psi_2^* \rangle\rangle}\; \mathsf{R2}}{\Sigma_0{}^*, \langle \Phi_1^* /\!/ {\Phi'_2}^*, \langle \Psi_1^*, B^\bullet /\!/ \Psi_2^*, C^\circ \rangle\rangle}\; \mathsf{m}}{\Sigma_0{}^*, \langle \Phi_1^* /\!/ {\Phi'_2}^*, \langle \Psi_1^* /\!/ \Psi_2^*, C^\circ \rangle\rangle}\; \mathrm{IH} \quad \dfrac{\Sigma_0{}^*, \langle \Phi_1^*, B \rightarrow C^\circ /\!/ {\Phi'_2}^*, \langle \Psi_1^* /\!/ \Psi_2^* \rangle\rangle \quad \dfrac{\overset{\vdots}{\Sigma_0, \langle B \rightarrow C^\bullet, \Phi_1 /\!/ B \rightarrow C^\bullet, \Phi'_2, \langle \Psi_1, /\!/ \Psi_2, C^\bullet \rangle\rangle}}{\Sigma_0, \langle B \rightarrow C^\bullet, \Phi_1 /\!/ \Phi'_2, \langle \Psi_1, /\!/ \Psi_2, C^\bullet \rangle\rangle}\; (\mathsf{lift})}{\Sigma_0, \langle \Phi_1 /\!/ \Phi'_2, \langle \Psi_1, /\!/ \Psi_2, C^\bullet \rangle\rangle}\; \mathrm{IH}}{\Sigma_0, \langle \Phi_1 /\!/ \Phi'_2, \langle \Psi_1 /\!/ \Psi_2 \rangle\rangle}\; \mathrm{IH}$$

This completes the proof. □

3 Consequence of cut-admissibility

In this section, we establish several consequences of the cut-admissibility result as applications of the calculus $\mathbf{C}_{\mathsf{B}^+}$. By means of the calculus, we provide syntactic proofs for several fundamental meta-logical properties like verification lemma and variable-sharing property. The former characterizes the relevant implication, which can also be proved by labeled calculus (cf. e.g., [8]); the latter, as a fundamental feature for relevant logic, however, is not obvious or even impossible to be proved by other existing proof systems like Hilbert calculus or labeled sequent calculus.

3.1 Verification lemma

In this subsection, we show that the verification lemma can be formalized syntactically in this calculus.

Let us consider the following rules:

$$\frac{\langle A^{\bullet}, B^{\circ} /\!/ - \rangle}{A \to B^{\circ}}\ \mathsf{V1} \qquad \frac{\langle - /\!/ A^{\bullet}, B^{\circ} \rangle}{A \to B^{\circ}}\ \mathsf{V2}$$

where double line in an inference rule means the premise and conclusion are interchangeable.

Notice that the two rules $\mathsf{V1}, \mathsf{V2}$ can be regarded the syntactic version of the verification lemma (cf. Lemma 1.7). This is due to the following fact, which is easily verified using the frame condition (F1).

Fact 3.1. *Let $\mathscr{M} = (\mathscr{O}, K, R, V)$ be a model. The following statements are equivalent:*

- $\langle A^{\bullet}, B^{\circ} /\!/ - \rangle$ *is valid in* $\mathscr{M}$*;*
- $\langle - /\!/ A^{\bullet}, B^{\circ} \rangle$ *is valid in* $\mathscr{M}$*;*
- *for every* $a \in K$*,* $a \Vdash A$ *and* $a \Vdash B$*.*

By Lemma 1.7, we have the following immediately:

Corollary 3.2. *The two rules* $\mathsf{V1}, \mathsf{V2}$ *are valid.*

The next proposition shows that the two rules above are admissible in $\mathbf{C}_{\mathsf{B}^+}$.

Proposition 3.3. *The two rules* $\mathsf{V1}, \mathsf{V2}$ *are admissible in* $\mathbf{C}_{\mathsf{B}^+}$.

Proof. We only show the admissibility of $\mathsf{V1}$ as the proof for $\mathsf{V2}$ is similar. First, assume $A \to B^{\circ}$ is derivable in $\mathbf{C}_{\mathsf{B}^+}$. Since $\mathbf{C}_{\mathsf{B}^+}$ is cut-free and $A \to B^{\circ}$ itself cannot be initial, we have $\langle A^{\bullet} /\!/ B^{\circ} \rangle$ derivable as well. Then, by $\mathsf{R1}$, we derive $\langle A^{\bullet}, B^{\circ} /\!/ - \rangle$. Conversely, assume $\langle A^{\bullet}, B^{\circ} /\!/ - \rangle$ is derivable, we have the following short derivation of $A \to B^{\circ}$:

$$\dfrac{\dfrac{\langle A^{\bullet}, B^{\circ} /\!/ - \rangle \qquad \dfrac{\langle A^{\bullet}, B^{\bullet} /\!/ B^{\bullet}, B^{\circ} \rangle}{\langle A^{\bullet}, B^{\bullet} /\!/ B^{\circ} \rangle}\ (\mathsf{lift})}{\langle A^{\bullet} /\!/ B^{\circ} \rangle}\ (\mathsf{cut})}{A \to B^{\circ}}\ \to^{\circ}$$

Since (cut) is admissible in $\mathbf{C}_{\mathsf{B}^+}$, we conclude $A \to B^{\circ}$ has a (cut)-free derivation. □

3.2 Variable-sharing property

The variable-sharing property is a common property among relevant logic and even regarded the key intrinsic feature of relevance [33]. In what follows, we provide a syntactic proof of the variable-sharing property of B^+ by means of the nested calculus $\mathbf{C}_{\mathsf{B}^+}$. This shows that the nested setting well captures the essence of relevance of the logic.

Definition 3.4 (Variable-sharing property)**.** A logic L is said to have the *variable-sharing property* (VSP for short) if whenever $A \to B$ is a theorem in L, $\mathsf{Var}(A) \cap \mathsf{Var}(B) \neq \varnothing$.

We show that a generalized form of variable-sharing property is preserved by backward rule applications of $\mathbf{C}_{\mathsf{B}^+}$. Before that, let us first define the input and output parts for a formula occurrence and then a nested sequent.

Definition 3.5. For a formula A occurring in a nested sequent Γ, we specify its input and output part as follows:

- if A occurs as $A^{\bullet}$, let $\mathsf{Input}(A^{\bullet}) = \{A\}$ and $\mathsf{Output}(A^{\bullet}) = \varnothing$

- if A occurs as A°, let

$$\mathsf{Input}(A^\circ) = \begin{cases} \varnothing & \text{if } A = p \\ \mathsf{Input}(B^\circ) \cup \mathsf{Input}(C^\circ) & \text{if } A = B@C \text{ where } @ \in \{\vee, \wedge\} \\ \mathsf{Input}(B^\bullet) & \text{if } A = B \to C \end{cases}$$

$$\mathsf{Output}(A^\circ) = \begin{cases} \{p\} & \text{if } A = p \\ \mathsf{Output}(B^\circ) \cup \mathsf{Output}(C^\circ) & \text{if } A = B@C \text{ where } @ \in \{\vee, \wedge\} \\ \mathsf{Output}(C^\circ) & \text{if } A = B \to C \end{cases}$$

The input (resp. output) part of Γ denoted $\mathsf{Input}(\Gamma)$ (resp. $\mathsf{Output}(\Gamma)$) is then defined as the collection of input (resp. output) parts of all the formulas occurring in Γ.

Remark 3.6. The general idea of input/output parts of a sequent is to specify the $\bullet$-part and the $\circ$-formula. However, as the rules $\to^\circ$ and $\to^\bullet$ change polarity of (sub-)formula occurrences from conclusion to premise, we need to take the changes into consideration. Therefore, in the case of $B \to C^\circ$, we regard $B^\bullet$ as input while only C° as output. In principle, there should be a similar treatment for $\to^\bullet$-formulas, however, as we will see in the proof of a key lemma below, we only make use of one premise of the rule $\to^\bullet$ to apply the inductive hypothesis where polarity of formulas is preserved from the conclusion.

Lemma 3.7. *Let Γ be a nested sequent. If Γ is provable in $\mathbf{C}_{\mathsf{B}^+}$, then we have* $\mathsf{Var}(\mathsf{Input}(\Gamma)) \cap \mathsf{Var}(\mathsf{Output}(\Gamma)) \neq \varnothing$.

Proof. By induction on the height h of the derivation of Γ. First, if $h = 0$, meaning that Γ is an axiom, so Γ is of the form $G\{\Sigma, p^\bullet, p^\circ\}$. Since $p \in \mathsf{Var}(\mathsf{Input}(\Gamma))$, the required condition holds trivially.

Next, if $h > 0$, let us consider the last rule r applied in the derivation and verify each rule of $\mathbf{C}_{\mathsf{B}^+}$ in turn. The cases regarding $\vee^\bullet$ and $\wedge^\bullet$ as well as (lift) are straightforward, we only show the following cases.

- $r = \vee^\circ$. Assume $\Gamma = G\{\Sigma, A \vee B^\circ\}$ and is derived from $\Gamma' = G\{\Sigma, A \vee B^\circ\}$. By definition, we have $\mathsf{Input}(\Gamma') = \mathsf{Input}(\Gamma)$, $\mathsf{Output}(\Gamma') = \mathsf{Output}(A^\circ)$, $\mathsf{Output}(\Gamma) = \mathsf{Output}(A \vee B^\circ) = \mathsf{Output}(A^\circ) \cup \mathsf{Output}(B^\circ)$, hence $\mathsf{Output}(\Gamma') \subseteq \mathsf{Output}(\Gamma)$, which means $\mathsf{Var}(\mathsf{Output}(\Gamma')) \subseteq \mathsf{Var}(\mathsf{Output}(\Gamma))$. By IH, $\mathsf{Var}(\mathsf{Input}(\Gamma')) \cap \mathsf{Var}(\mathsf{Input}(\Gamma')) \neq \varnothing$, thus $\mathsf{Var}(\mathsf{Input}(\Gamma)) \cap \mathsf{Var}(\mathsf{Input}(\Gamma')) \neq \varnothing$.
- $r = \wedge^\circ$. Similar to the case of $\vee^\circ$.
- $r = \to^\bullet$. Assume $\Gamma = G\{\Sigma, A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2 \rangle\}$ and is derived from the two premises $\Gamma_1 = G^*\{\Sigma^*, A \to B^\bullet, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle\}$ and $\Gamma_2 = G\{\Sigma, A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2, B^\bullet \rangle\}$. By definition, we have $\mathsf{Input}(\Gamma_2) = \mathsf{Input}(\Gamma) \cup \mathsf{Input}(B^\bullet)$. Since $A \to B \in \mathsf{Input}(\Gamma)$, it follows that $\mathsf{Var}(\mathsf{Input}(\Gamma_2)) = \mathsf{Var}(\mathsf{Input}(\Gamma))$. Meanwhile, obviously $\mathsf{Var}(\mathsf{Output}(\Gamma_2)) = \mathsf{Var}(\mathsf{Output}(\Gamma))$. By IH, we have $\mathsf{Var}(\mathsf{Input}(\Gamma_2)) \cap \mathsf{Var}(\mathsf{Input}(\Gamma_2)) \neq \varnothing$, thus $\mathsf{Var}(\mathsf{Input}(\Gamma)) \cap \mathsf{Var}(\mathsf{Input}(\Gamma')) \neq \varnothing$.
- $r = \to^\circ$. Assume $\Gamma = G\{\Sigma, A \to B^\circ\}$ and is derived from $\Gamma' = G\{\Sigma, \langle A^\bullet /\!/ B^\circ \rangle\}$. By definition, $\mathsf{Input}(\Gamma') = \mathsf{Input}(\Gamma) = \mathsf{Input}(G\{\Sigma\}) \cup \mathsf{Input}(A^\bullet) \cup \mathsf{Input}(B^\circ)$ and $\mathsf{Output}(\Gamma') = \mathsf{Output}(\Gamma) = \mathsf{Output}(B^\circ)$. Hence $\mathsf{Var}(\mathsf{Output}(\Gamma')) = \mathsf{Var}(\mathsf{Output}(\Gamma))$. By IH, we have $\mathsf{Var}(\mathsf{Input}(\Gamma')) \cap \mathsf{Var}(\mathsf{Input}(\Gamma')) \neq \varnothing$, thus $\mathsf{Var}(\mathsf{Input}(\Gamma)) \cap \mathsf{Var}(\mathsf{Input}(\Gamma')) \neq \varnothing$.

This completes the proof. □

Theorem 3.8 (VSP for B^+). *If $A \to B$ is a theorem in B^+, then* $\mathsf{Var}(A) \cap \mathsf{Var}(B) \neq \varnothing$.

Proof. Since $A \to B$ is a theorem in B^+ and $\mathbf{C}_{\mathsf{B}^+}$ is complete, it follows $A \to B^\circ$ is derivable in $\mathbf{C}_{\mathsf{B}^+}$. Given that the calculus is cut-free, meaning that $A \to B^\circ$ is only derivable from $\langle A^\bullet \mathbin{/\!/} B^\circ \rangle$. By Lemma 3.7, we have $\mathsf{Var}(A) \cap \mathsf{Var}(B) \neq \varnothing$. $\square$

A Appendix

Theorem 2.11 (Completeness of $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$)**.** *All the axioms and rules in $\mathscr{H}_{\mathsf{B}^+}$ are derivable in $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$. Consequently, if a formula $A \in \mathsf{Frm}$ is valid, then it is provable in $\mathbf{C}_{\mathsf{B}^+} + (\mathsf{cut})$.*

Proof. We proceed to derive the axioms and rules of $\mathbf{C}_{\mathsf{B}^+}$. Since the derivations are routine, we restrict attention to a few illustrative cases.

$\vdash_{\mathbf{C}_{\mathsf{B}^+}} A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)^\circ$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\langle A^\bullet, B^\bullet /\!/ A^\bullet, A^\circ\rangle}\,(\mathsf{id})}{\langle A^\bullet, B^\bullet /\!/ A^\circ\rangle}\,(\mathsf{lift}) \quad \dfrac{\dfrac{}{\langle A^\bullet, B^\bullet /\!/ B^\bullet, B^\circ\rangle}\,(\mathsf{id})}{\langle A^\bullet, B^\bullet /\!/ B^\circ\rangle}\,(\mathsf{lift})}{\langle A^\bullet, B^\bullet /\!/ A \wedge B^\circ\rangle}\,\wedge^\circ}{\langle A^\bullet, B^\bullet /\!/ (A \wedge B) \vee (A \wedge C)^\circ\rangle}\,\vee^\circ \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\langle A^\bullet, C^\bullet /\!/ A^\bullet, A^\circ\rangle}\,(\mathsf{id})}{\langle A^\bullet, C^\bullet /\!/ A^\circ\rangle}\,(\mathsf{lift}) \quad \dfrac{\dfrac{}{\langle A^\bullet, C^\bullet /\!/ C^\bullet, C^\circ\rangle}\,(\mathsf{id})}{\langle A^\bullet, C^\bullet /\!/ C^\circ\rangle}\,(\mathsf{lift})}{\langle A^\bullet, C^\bullet /\!/ A \wedge C^\circ\rangle}\,\wedge^\circ}{\langle A^\bullet, C^\bullet /\!/ (A \wedge B) \vee (A \wedge C)^\circ\rangle}\,\vee^\circ}{\dfrac{\langle A^\bullet, B \vee C^\bullet /\!/ (A \wedge B) \vee (A \wedge C)^\circ\rangle}{\dfrac{\langle A \wedge (B \vee C)^\bullet /\!/ (A \wedge B) \vee (A \wedge C)^\circ\rangle}{A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)^\circ}\,\to^\circ}\,\wedge^\bullet}\,\vee^\bullet$$

For mp, we show that if $\vdash_{\mathbf{C}_{\mathsf{B}^+}} A \to B^\circ$ and $\vdash_{\mathbf{C}_{\mathsf{B}^+}} A^\circ$, then $\vdash_{\mathbf{C}_{\mathsf{B}^+}} B^\circ$.

$$\dfrac{A^\circ \qquad \dfrac{\dfrac{A \to B^\circ \qquad \dfrac{\dfrac{}{A \to B^\bullet, \langle A^\bullet, A^\circ /\!/ -\rangle}\,(\mathsf{id}) \quad \dfrac{}{A \to B^\bullet, \langle A^\bullet /\!/ B^\bullet, B^\circ\rangle}\,(\mathsf{id})}{A \to B^\bullet, \langle A^\bullet /\!/ B^\circ\rangle}\,\to^\bullet}{\langle A^\bullet /\!/ B^\circ\rangle}\,(\mathsf{cut})}{A^\bullet, B^\circ}\,\mathsf{R1}}{B^\circ}\,(\mathsf{cut})$$

For aff, we prove that if $\vdash_{\mathbf{C}_{\mathsf{B}^+}} A \to B^\circ$ and $\vdash_{\mathbf{C}_{\mathsf{B}^+}} C \to D^\circ$, then $\vdash_{\mathbf{C}_{\mathsf{B}^+}} (B \to C) \to (A \to D)^\circ$.

$$\dfrac{\dfrac{\dfrac{A \to B^\circ \qquad \dfrac{\dfrac{}{A \to B^\bullet, \langle A^\bullet, A^\circ /\!/ -\rangle}\,(\mathsf{id}) \quad \dfrac{}{A \to B^\bullet, \langle A^\bullet /\!/ B^\bullet, B^\circ\rangle}\,(\mathsf{id})}{A \to B^\bullet, \langle A^\bullet /\!/ B^\circ\rangle}\,\to^\bullet}{\langle A^\bullet /\!/ B^\circ\rangle}\,(\mathsf{cut})}{\langle B \to C^\bullet, \langle A^\bullet, B^\circ /\!/ -\rangle /\!/ -\rangle}\,\mathsf{R1} \qquad \dfrac{\dfrac{C \to D^\circ \qquad \dfrac{\dfrac{}{C \to D^\bullet, \langle C^\bullet, C^\circ /\!/ -\rangle}\,(\mathsf{id}) \quad \dfrac{}{C \to D^\bullet, \langle C^\bullet /\!/ D^\bullet, D^\circ\rangle}\,(\mathsf{id})}{C \to D^\bullet, \langle C^\bullet /\!/ D^\circ\rangle}\,\to^\bullet}{\langle C^\bullet /\!/ D^\circ\rangle}\,(\mathsf{cut})}{\langle B \to C^\bullet, \langle A^\bullet /\!/ C^\bullet, D^\circ\rangle /\!/ -\rangle}\,\mathsf{R1}}{\dfrac{\langle B \to C^\bullet, \langle A^\bullet /\!/ D^\circ\rangle /\!/ -\rangle}{\dfrac{\langle B \to C^\bullet /\!/ \langle A^\bullet /\!/ D^\circ\rangle\rangle}{\dfrac{\langle B \to C^\bullet /\!/ A \to D^\circ\rangle}{(B \to C) \to (A \to D)^\circ}\,\to^\circ}\,\to^\circ}\,\mathsf{R2}}\,\to^\bullet$$

This completes the proof. □

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